MULTIPLE SOLUTIONS TO LOGARITHMIC DOUBLE PHASE PROBLEMS INVOLVING SUPERLINEAR NONLINEARITIES

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Abstract. This paper investigates a class of problems involving a logarithmic double phase operator with variable exponents and right-hand sides that consist of nonlinearities exhibiting subcritical and superlinear growth. Under very general assumptions we prove the existence of at least two nontrivial bounded weak solutions for such problems whereby the solutions have opposite energy sign. In addition, we give conditions on the nonlinearity under which the solutions turn out to be nonnegative.

1. Introduction

During the last decade, problems with unbalanced growth have become more important. These problems are generally characterized by operators that are of the form

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right), \tag{1.1}$$

with the corresponding energy functional

$$u\mapsto\int_\Omega\left(\frac{|\nabla u|^{p(x)}}{p(x)}+\mu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right)\mathrm{d}x. \tag{1.2}$$

Such type of functionals appeared for the first time for constant exponents as

$$J(u)=\int_\Omega\left(|\nabla u|^p+\mu(x)|\nabla u|^q\right)\mathrm{d}x \tag{1.3}$$

in the work of Zhikov [57] related to homogenization and elasticity theory. Indeed, the coefficient μ corresponds to the geometry of composites made of two materials of hardness p and q. It should be noted that functionals of the form (1.3) are special cases of the groundbreaking works by Marcellini [39, 40] which are related to more general problems with nonstandard growth and p,q-growth conditions, see also the more recent works by Cupini–Marcellini–Mascolo [24] and Marcellini [37, 38]. Later on, the results of Marcellini in the concrete setting of double phase integrals given by (1.3) have been improved by the pioneering papers by Baroni–Colombo–Mingione [7, 8, 9] and Colombo–Mingione [21, 22], see also Ragusa–Tachikawa [46] for studying (1.2) and Chems Eddine–Ouannasser–Ragusa [15] for the anisotropic case. Furthermore, double phase operators as in (1.1) (also for p, q being constants) occur more frequently not only in the mathematical sense,

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but also in applications. In this direction, we mention the works by Bahrouni–Rădulescu–Repovš [6] on transonic flows, Benci–D'Avenia–Fortunato–Pisani [10] on quantum physics, Cherfils–Il'yasov [17] for reaction diffusion systems and Zhikov [58, 59] on the Lavrentiev gap phenomenon, the thermistor problem and the duality theory. For a comprehensive overview of the main properties of the related function space and the double phase operator, we direct the reader to the papers by Colasuonno–Perera [19], Colasuonno–Squassina [20], Crespo-Blanco–Gasiński–Harjulehto–Winkert [23], Ho–Winkert [33], Liu–Dai [34], and Perera–Squassina [44].

Recently, Arora–Crespo-Blanco–Winkert [5] introduced and study a new operator, called logarithmic double phase operator, of the form

$$\begin{aligned}\operatorname{div} \mathcal{A}(u) = \operatorname{div} \Bigg(& |\nabla u|^{p(x)-2}\nabla u \\ & + \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right] |\nabla u|^{q(x)-2}\nabla u\Bigg),\end{aligned} \tag{1.4}$$

whereby the function u belongs to an appropriate Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ generated by the function

$$\mathcal{H}_{\log}(x,t) = t^{p(x)} + \mu(x)t^{q(x)}\log(e+t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty),$$

for $p, q \in C(\overline{\Omega})$ with $1 < p(x) < N$, $p(x) < q(x)$ for all $x \in \overline{\Omega}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$. The operator (1.4) extends the classical double phase operator by incorporating logarithmic terms and this generalization enables us to account for not only power-law growth in each term but also other growth behaviors, especially those involving logarithmic functions. As a result, it leads to nonuniform ellipticity of the energy functional related to (1.4) given by

$$u \to \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)} \log(e+|\nabla u|)\right) \mathrm{d}x \tag{1.5}$$

presenting additional analytical challenges. In particular, the inclusion of logarithmic terms introduces a modulation effect, which is essential in modeling inhomogeneous materials with spatially varying structural properties. We also point out that functionals of the form (1.5) have been investigated in several works for special cases of p and q. Baroni–Colombo–Mingione [8] proved the local Hölder continuity of the gradient of local minimizers of

$$u \mapsto \int_\Omega \left[|\nabla u|^p + \mu(x)|\nabla u|^p \log(e+|\nabla u|)\right] \mathrm{d}x,$$

provided $1 < p < \infty$ and $0 \leq \mu(\cdot) \in C^{0,\alpha}(\overline{\Omega})$ while De Filippis–Mingione [26] showed the local Hölder continuity of the gradients of local minimizers of the functional

$$u \mapsto \int_\Omega \left[|\nabla u| \log(1+|\nabla u|) + \mu(x)|\nabla u|^q\right] \mathrm{d}x, \tag{1.6}$$

whenever $0 \leq \mu(\cdot) \in C^{0,\alpha}(\overline{\Omega})$ and $1 < q < 1 + \frac{\alpha}{n}$. Note that functionals of the form (1.6) originate from functionals with nearly linear growth given by

$$u \mapsto \int_\Omega |\nabla u| \log(1+|\nabla u|)\, \mathrm{d}x, \tag{1.7}$$

see the works by Fuchs–Mingione [29] and Marcellini–Papi [41]. We also mention that functionals as in (1.7) appear in the theory of plasticity with logarithmic hardening, see Seregin–Frehse [49] and Fuchs–Seregin [30]. Also the work of Marcellini [39] includes logarithmic functions defined by

$$u \mapsto \int_\Omega (1+|\nabla u|^2)^{\frac{p}{2}} \log(1+|\nabla u|)\,\mathrm{d}x.$$

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we consider the following parametric Dirichlet problem

$$-\operatorname{div} \mathcal{A}(u) = \lambda f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

where $\operatorname{div}\mathcal{A}$ is the logarithmic double phase operator given in (1.4) and $\lambda > 0$ is a parameter to be specified. In the following, we denote by κ the constant given by

$$\kappa = \frac{e}{e+t_0}, \tag{1.9}$$

where e is Euler's number and t_0 is the unique positive number that satisfies $t_0 = e\log(e+t_0)$. First, we define

$$C_+(\overline{\Omega}) = \left\{ r \in C(\overline{\Omega}) \colon 1 < r(x) \text{ for all } x \in \overline{\Omega} \right\}$$

and set, for any $r \in C_+(\overline{\Omega})$,

$$r_- := \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r_+ := \max_{x\in\overline{\Omega}} r(x).$$

We assume the following hypotheses on the data:

(H) $p,q \in C_+(\overline{\Omega})$ such that $p(x) < N$, $p(x) < q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$ and $\mu \in L^\infty(\Omega)$ with $\mu(\cdot) \geq 0$.

($\mathrm{H_f}$) Let $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $F(x,t) = \int_0^t f(x,\xi)\,\mathrm{d}\xi$ be such that:

- ($\mathrm{f_1}$) The function f is of Carathéodory type, i.e., $t \mapsto f(x,t)$ is continuous for a.a. $x \in \Omega$ and $x \mapsto f(x,t)$ is measurable for all $t \in \mathbb{R}$.
- ($\mathrm{f_2}$) There exist $s \in C_+(\overline{\Omega})$ with $s_+ < (p_-)^*$ and $C > 0$ such that
$$|f(x,t)| \leq C\left(1 + |t|^{s(x)-1}\right)$$
for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$.
- ($\mathrm{f_3}$)
$$\lim_{\xi \to \pm\infty} \frac{F(x,\xi)}{|\xi|^{q_+}\log(e+|\xi|)} = +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$
- ($\mathrm{f_4}$) There exist $\ell, \widetilde{\ell} \in C_+(\overline{\Omega})$ such that $\min\{\ell_-, \widetilde{\ell}_-\} \in \left((s_+ - p_-)\frac{N}{p_-}, s_+\right)$ and $K > 0$ with
$$0 < K \leq \liminf_{\xi \to +\infty} \frac{f(x,\xi)\xi - q_+\left(1+\frac{\kappa}{q_-}\right)F(x,\xi)}{|\xi|^{\ell(x)}}$$
uniformly for a.a. $x \in \Omega$, and
$$0 < K \leq \liminf_{\xi \to -\infty} \frac{f(x,\xi)\xi - q_+\left(1+\frac{\kappa}{q_-}\right)F(x,\xi)}{|\xi|^{\widetilde{\ell}(x)}}$$

uniformly for a.a. $x \in \Omega$, where κ is given by (1.9).

Our first result reads as follows.

Theorem 1.1. *Let hypotheses* (H) *and* ($\mathrm{H_f}$) *be satisfied and suppose that there exist* $r, \eta > 0$ *such that*

$$\max\left\{\eta^{p_-}, \eta^{q_+} \log\left(e + \frac{2\eta}{R}\right)\right\} < \delta r \tag{1.10}$$

such that

($\mathrm{h_1}$) $F(x,t) \geq 0$ *for a.a.* $x \in \Omega$ *and for all* $t \in [0, \eta]$,
($\mathrm{h_2}$) $\alpha(r) < \beta(\eta)$,

where $\alpha(r)$ *and* $\beta(\eta)$ *are defined in* (3.7) *and* (3.8)*, respectively. Then, for each* $\lambda \in \Lambda$*, with*

$$\Lambda := \left(\frac{1}{\beta(\eta)}, \frac{1}{\alpha(r)}\right),$$

problem (1.8) *admits at least two nontrivial bounded weak solutions which have opposite energy sign.*

If f is in addition nonnegative and has a special behavior near the origin, we obtain the following result.

Theorem 1.2. *Let hypotheses* (H) *and* ($\mathrm{H_f}$) *be satisfied and suppose that the nonlinearity* f *is nonnegative and fulfills*

$$\limsup_{t \to 0^+} \frac{\inf_{x \in \Omega} F(x,t)}{t^{p_-}} = +\infty. \tag{1.11}$$

Then, for each $\lambda \in (0, \lambda^*)$*, where*

$$\lambda^* = \sup_{r>0} \frac{1}{\alpha(r)},$$

with $\alpha(r)$ *given in* (3.7)*, problem* (1.8) *admits at least two nontrivial, nonnegative, bounded weak solutions which have opposite energy sign.*

The proofs of Theorem 1.1 and 1.2 are based on a critical point result due to Bonanno–D'Aguì [12] which applies to more general classes of variational problems. Furthermore, we give a concrete interval to which the solutions belong. Our paper can be seen as an extension of the works by Chinnì–Sciammetta–Tornatore [18], Sciammetta–Tornatore [47] and Amoroso–Bonanno–D'Aguì–Winkert [1]. The differences to [18] are twofold: first, in [18] the operator is the well-known $(q(\cdot), p(\cdot))$-Laplacian and so the function space is the usual Sobolev space $W_0^{1,q(\cdot)}(\Omega)$ while we are, in addition, able to weaken the assumptions on f in our paper not supposing the usual Ambrosetti-Rabinowitz condition. This condition says that there exist $\mu > q_+$ and $M > 0$ such that

$$0 < \mu F(x,s) \leq f(x,s)s \tag{AR}$$

for a.a. $x \in \Omega$ and for all $|s| \geq M$. Instead of condition (AR) we suppose that the primitive of f is q-superlinear at $\pm\infty$ with a logarithmic term (see ($\mathrm{f_3}$)) along with another behavior near $\pm\infty$, see ($\mathrm{f_4}$). Both conditions are weaker than (AR). Note that we do not need any behavior of f or its primitive near the origin in Theorem

1.1. As mentioned above the abstract critical point theorem we used is due to Bonanno–D'Aguì [12] and was applied in the same paper to the p-Laplace problem

$$-\Delta_p u = \lambda f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.12}$$

in order to get two nontrivial solutions of (1.12).

As already mentioned, the operator in (1.4) has been recently introduced in the work by Arora–Crespo-Blanco–Winkert [5] who studied problems of the form

$$-\operatorname{div} \mathcal{A}(u) = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.13}$$

where $\operatorname{div} \mathcal{A}$ is as in (1.4) and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that has subcritical growth fulfilling appropriate conditions. Based on the Nehari manifold treatment, the existence of a sign-changing solution of (1.13) has been shown under the more strict assumption that $q_+ + 1 < (p_-)^*$, see also the recent work by the same authors [4] related to more general embeddings and existence results based on the concentration compactness principle. Moreover, Lu–Vetro–Zeng [36] studied existence and uniqueness of equations involving the operator

$$u \mapsto \Delta_{\mathcal{H}_L} u = \operatorname{div}\left(\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u\right), \quad u \in W^{1,\mathcal{H}_L}(\Omega), \tag{1.14}$$

where $\mathcal{H}_L\colon \Omega \times \times[0,\infty) \to [0,\infty)$ is given by

$$\mathcal{H}_L(x,t) = [t^{p(x)} + \mu(x)t^{q(x)}]\log(e+\alpha t),$$

with $\alpha \geq 0$, see also Vetro–Zeng [52]. We point out that the operator (1.14) is different from the one in (1.4). In this direction, we also mention the paper by Vetro–Winkert [51] who proved the existence of a solution to the logarithmic problem with convection term of the form

$$-\operatorname{div} \mathcal{A}(u) = f(x,u,\nabla u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.15}$$

where $\operatorname{div} \mathcal{A}$ is as in (1.4) and $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function that satisfies certain growth and coercivitiy conditions. The authors prove the boundedness, closedness and compactness of the corresponding solution set to (1.15), see also the recent work by Vetro [50] concerning related Kirchhoff problems. All in all, our paper extends the works by Amoroso–Bonanno–D'Aguì–Winkert [1], Chinnì–Sciammetta–Tornatore [18], Sciammetta–Tornatore [47], and Sciammetta–Tornatore–Winkert [48] to more general operators and under weaker assumptions, see also related problems with Neumann or Robin boundary conditions in the papers by Amoroso–Crespo-Blanco–Pucci–Winkert [2], Amoroso–Morabito [3] or D'Aguì–Sciammetta–Tornatore–Winkert [25]. Finally, we also mention some important works dealing with double phase problems and different assumptions on the right-hand side, see the papers by Biagi–Esposito–Vecchi [11], Borer–Pimenta–Winkert [13], Bouaam–El Ouaarabi–Melliani [14], Cheng–Shang–Bai [16], Gasiński–Winkert [31], Liu-Pucci [35], Papageorgiou–Rădulescu–Repovš [42], Zeng–Bai–Gasiński–Winkert [53], Zeng–Rădulescu–Winkert [54, 55], Zhang–Rădulescu [56] and the references therein.

The paper is organized as follows. In Section 2 we present the main properties of variable exponent Sobolev spaces and Musielak-Orlicz Sobolev spaces with logarithmic perturbation as well as the properties of the logarithmic double phase operator. Also, we recall a general critical point theorem which is the basis of our treatment. Finally, in Section 3 we give the proofs of our main results by applying variational and topological tools.

2. Preliminaries and variational framework

In this section we recall the main properties of variable exponent Sobolev spaces and Musielak-Orlicz Sobolev spaces with logarithmic perturbation. We also present the main properties of the logarithmic double phase operator and mention some tools which will be needed in the sequel. The results are primarily taken from the monographs by Diening–Harjulehto–Hästö–Růžička [27] , Harjulehto–Hästö [32], Papageorgiou–Winkert [43] as well as the paper by Arora–Crespo-Blanco–Winkert [5], Crespo-Blanco–Gasiński–Harjulehto–Winkert [23] and Fan–Zhao [28].

To this end, let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For $1 \leq r \leq \infty$, we denote by $L^r(\Omega)$ the usual Lebesgue spaces equipped with the standard norm $\|\cdot\|_r$ and for $1 \leq r < \infty$ the corresponding Sobolev spaces $W^{1,r}(\Omega)$ and $W^{1,r}_0(\Omega)$ equipped with the usual norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0} = \|\nabla \cdot\|_r$, respectively. Denoting by $M(\Omega)$ the set of all measurable functions $u\colon \Omega \to \mathbb{R}$, for any $r \in C_+(\overline{\Omega})$ we introduce the Lebesgue space $L^{r(\cdot)}(\Omega)$ with variable exponent by

$$L^{r(\cdot)}(\Omega) = \left\{u \in M(\Omega)\colon \varrho_{r(\cdot)}(u) < \infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_{r(\cdot)} = \inf\left\{\tau > 0 : \varrho_{r(\cdot)}\left(\frac{u}{\tau}\right) \leq 1\right\},$$

where the corresponding modular is given by

$$\varrho_{r(\cdot)}(u) := \int_\Omega |u|^{r(x)}\,\mathrm{d}x.$$

It is well known that the space $L^{r(\cdot)}(\Omega)$ is a separable and reflexive Banach space with a uniformly convex norm. Its dual space is given by $\left[L^{r(\cdot)}(\Omega)\right]^* = L^{r'(\cdot)}(\Omega)$, where $r'(\cdot)$ denotes the conjugate variable exponent of $r(\cdot)$, that is,

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Moreover, a weaker version of Hölder's inequality holds in these spaces, stating that

$$\int_\Omega |uv|\,\mathrm{d}x \leq \left[\frac{1}{r_-} + \frac{1}{r_+}\right] \|u\|_{r(\cdot)}\|v\|_{r'(\cdot)} \leq 2\|u\|_{r(\cdot)}\|v\|_{r'(\cdot)}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and $v \in L^{r'(\cdot)}(\Omega)$. Additionally, if $r_1, r_2 \in C_+(\overline{\Omega})$ satisfy $r_1(x) \leq r_2(x)$ for all $x \in \overline{\Omega}$, then the continuous embedding

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$$

holds. Next, we recall the following proposition, which establishes a relation between the norm and its associated modular function, see the paper by Fan–Zhao [28] for its proof.

Proposition 2.1. *Let $r \in C_+(\overline{\Omega})$, $u \in L^{r(\cdot)}(\Omega)$ and $\tau > 0$. Then the following hold:*

(i) *If $u \neq 0$, then $\|u\|_{r(\cdot)} = \tau \iff \varrho_{r(\cdot)}(\frac{u}{\tau}) = 1$;*
(ii) *$\|u\|_{r(\cdot)} < 1$ (resp. > 1, $= 1$) $\iff \varrho_{r(\cdot)}(u) < 1$ (resp. > 1, $= 1$);*
(iii) *If $\|u\|_{r(\cdot)} < 1 \implies \|u\|^{r_+}_{r(\cdot)} \leq \varrho_{r(\cdot)}(u) \leq \|u\|^{r_-}_{r(\cdot)}$;*
(iv) *If $\|u\|_{r(\cdot)} > 1 \implies \|u\|^{r_-}_{r(\cdot)} \leq \varrho_{r(\cdot)}(u) \leq \|u\|^{r_+}_{r(\cdot)}$;*
(v) *$\|u\|_{r(\cdot)} \to 0 \iff \varrho_{r(\cdot)}(u) \to 0$;*

(vi) $\|u\|_{r(\cdot)} \to +\infty \iff \varrho_{r(\cdot)}(u) \to +\infty$.

Starting from the Lebesgue space $L^{r(\cdot)}(\Omega)$, we define the variable exponent Sobolev space $W^{1,r(\cdot)}(\Omega)$ by

$$W^{1,r(\cdot)}(\Omega) = \left\{ u \in L^{r(\cdot)}(\Omega) : |\nabla u| \in L^{r(\cdot)}(\Omega) \right\},$$

equipped with the usual norm

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)},$$

where $\|\nabla u\|_{r(\cdot)} = \| \, |\nabla u| \, \|_{r(\cdot)}$. Moreover, we denote by $W_0^{1,r(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,r(\cdot)}(\Omega)$. We know that $W^{1,r(\cdot)}(\Omega)$ and $W_0^{1,r(\cdot)}(\Omega)$ are uniformly convex, separable and reflexive Banach spaces. In particular we know that a Poincaré inequality holds in the space $W_0^{1,r(\cdot)}(\Omega)$ and so, we can equip $W_0^{1,r(\cdot)}(\Omega)$ with the equivalent norm given by

$$\|u\|_{1,r(\cdot),0} = \|\nabla u\|_{r(\cdot)}.$$

Next, supposing hypothesis (H), we introduce the nonlinear function $\mathcal{H}_{\log}\colon \Omega \times [0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{H}_{\log}(x,t) = t^{p(x)} + \mu(x) t^{q(x)} \log(e+t) \quad \text{for all } (x,t) \in \Omega \times [0,\infty),$$

where e stands for Euler's number. Note that $\mathcal{H}_{\log}$ is measurable in the first variable, $\mathcal{H}_{\log}(x,0) = 0$ and $\mathcal{H}_{\log}(x,0) > 0$ for all $t > 0$. Moreover, $\mathcal{H}_{\log}$ satisfies the Δ_2-condition, that is

$$\mathcal{H}_{\log}(x,2t) \le M \mathcal{H}_{\log}(x,t)$$

for a.a. $x \in \Omega$, for all $t \in (0,+\infty)$ and for some $M \ge 2$. Then, we can introduce the corresponding Musielak-Orlicz space $L^{\mathcal{H}_{\log}}(\Omega)$ defined as

$$L^{\mathcal{H}_{\log}}(\Omega) = \left\{ u \in M(\Omega) \colon \varrho_{\mathcal{H}_{\log}}(u) < +\infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}_{\log}} = \inf \left\{ \alpha > 0 \colon \varrho_{\mathcal{H}_{\log}}\left(\frac{u}{\alpha}\right) \le 1 \right\},$$

where $\varrho_{\mathcal{H}_{\log}}(\cdot)$ is the corresponding modular, namely

$$\varrho_{\mathcal{H}_{\log}}(u) = \int_\Omega \mathcal{H}_{\log}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)} \log(e+|u|) \right) \mathrm{d}x.$$

The next proposition, whose proof can be found in [5, Proposition 3.4], establishes that $L^{\mathcal{H}_{\log}}(\Omega)$ is a separable and reflexive Banach space and provides the relation between the norm and the corresponding modular.

Proposition 2.2. *Let hypothesis* (H) *be satisfied. Then $L^{\mathcal{H}_{\log}}(\Omega)$ is a separable, reflexive Banach space and the following hold:*

(i) $\|u\|_{\mathcal{H}_{\log}} = \alpha \iff \varrho_{\mathcal{H}_{\log}}\left(\frac{u}{\alpha}\right) = 1$ *for* $u \neq 0$ *and* $\alpha > 0$*;*

(ii) $\|u\|_{\mathcal{H}_{\log}} < 1$ *(resp.* > 1, $= 1$*)* $\iff \varrho_{\mathcal{H}_{\log}}\left(\frac{u}{\alpha}\right) < 1$ *(resp.* > 1, $= 1$*);*

(iii) $\min\left\{ \|u\|_{\mathcal{H}_{\log}}^{p_-}, \|u\|_{\mathcal{H}_{\log}}^{q_+ + \kappa} \right\} \le \varrho_{\mathcal{H}_{\log}}(u) \le \max\left\{ \|u\|_{\mathcal{H}_{\log}}^{p_-}, \|u\|_{\mathcal{H}_{\log}}^{q_+ + \kappa} \right\}$ *where* $\kappa = \frac{e}{e+t_0}$ *is as in* (1.9)*;*

(v) $\|u\|_{\mathcal{H}_{\log}} \to 0 \iff \varrho_{\mathcal{H}_{\log}}(u) \to 0$*;*

(vi) $\|u\|_{\mathcal{H}_{\log}} \to +\infty \iff \varrho_{\mathcal{H}_{\log}}(u) \to +\infty$.

The following lemma will be used later.

Lemma 2.3. *Let $Q>1$ and $h\colon [0,\infty)\to[0,\infty)$ given by $h(t)=\frac{t}{Q(e+t)\log(e+t)}$. Then h attains its maximum value at t_0 and the value is $\frac{\kappa}{Q}$, where t_0 and κ are the same as in (1.9).*

Proof. It holds

$$h'(t)=\frac{(e+t)\log(e+t)-t(\log(e+t)+1)}{Q((e+t)\log(e+t))^2}=\frac{e\log(e+t)-t}{Q((e+t)\log(e+t))^2}.$$

Since the denominator is positive, $h'(t)=0$ is equivalent to $e\log(e+t)-t=0$. Denoting $g(t):=e\log(e+t)-t$, we see that $g(\cdot)$ is strictly decreasing for all $t>0$, $g(0)=e>0$ and $\lim_{t\to\infty}g(t)=-\infty$. Thus, g crosses zero exactly once and so there is a unique $t_0>0$ such that $e\log(e+t_0)=t_0$. Since $g(t)>0$ for $t<t_0$ and $g(t)<0$ for $t>t_0$, we obtain $h'(t)>0$ on $(0,t_0)$ and $h'(t)<0$ on (t_0,∞). Moreover, $h(0)=0$ and $\lim_{t\to\infty}h(t)=0$. Therefore, h has a strict global maximum at this unique t_0. The maximal value is, due to $t_0=e\log(e+t_0)$,

$$h(t_0)=\frac{t_0}{Q(e+t_0)\log(e+t_0)}=\frac{e\log(e+t_0)}{Q(e+t_0)\log(e+t_0)}=\frac{e}{Q(e+t_0)}=\frac{\kappa}{Q},$$

see (1.9). □

Next, we introduce the corresponding Musielak-Orlicz Sobolev space given by

$$W^{1,\mathcal{H}_{\log}}(\Omega)=\left\{u\in L^{\mathcal{H}_{\log}}(\Omega)\colon |\nabla u|\in L^{\mathcal{H}_{\log}}(\Omega)\right\},$$

equipped with the usual norm

$$\|u\|_{1,\mathcal{H}_{\log}}=\|\nabla u\|_{\mathcal{H}_{\log}}+\|u\|_{\mathcal{H}_{\log}},$$

where $\|\nabla u\|_{\mathcal{H}_{\log}}=\|\,|\nabla u|\,\|_{\mathcal{H}_{\log}}$. Further, we define

$$W_0^{1,\mathcal{H}_{\log}}(\Omega)=\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{H}_{\log}}}.$$

Here we recall Proposition 3.6 by Arora–Crespo-Blanco–Winkert [5] where the authors prove that $W^{1,\mathcal{H}_{\log}}(\Omega)$ and $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ are separable, reflexive Banach spaces. Note that $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ can be equipped with the equivalent norm

$$\|u\|=\|\nabla u\|_{\mathcal{H}_{\log}}\quad\text{for all } u\in W_0^{1,\mathcal{H}_{\log}}(\Omega),$$

see Arora–Crespo-Blanco–Winkert [5, Proposition 3.9] .

The next proposition states the main embedding results for these spaces, see again [5, Proposition 3.7].

Proposition 2.4. *Let (H) be satisfied, then the following embeddings hold:*

(i) *$W_0^{1,\mathcal{H}_{\log}}(\Omega)\hookrightarrow W_0^{1,p(\cdot)}(\Omega)$ is continuous;*
(ii) *$W_0^{1,\mathcal{H}_{\log}}(\Omega)\hookrightarrow L^{m(\cdot)}(\Omega)$ is compact for $m\in C(\overline{\Omega})$ with $1\le m(x)<p^*(x)$ for all $x\in\overline{\Omega}$;*
(iii) *$W^{1,\mathcal{H}_{\log}}(\Omega)\hookrightarrow L^{\mathcal{H}_{\log}(\cdot)}(\Omega)$ is compact.*

In particular, condition (ii) in Proposition 2.4 implies that there exists a constant $k_m>0$ such that

$$\|u\|_{m(\cdot)}\le k_m\|u\|. \tag{2.1}$$

Next, we introduce the nonlinear operator $A\colon W_0^{1,\mathcal{H}_{\log}}(\Omega)\to W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$ defined by

$$\begin{aligned}
&\langle A(u),v\rangle\\
&=\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x\\
&\quad+\int_\Omega \mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right)|\nabla u|^{q(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x,
\end{aligned}\tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$. The properties of $A\colon W_0^{1,\mathcal{H}_{\log}}(\Omega)\to W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$ are summarized in the following proposition, see Arora–Crespo-Blanco–Winkert [5, Theorem 4.4].

Theorem 2.5. *Let hypotheses* (H) *be satisfied and A be given as in* (2.2). *Then A is bounded, continuous, strictly monotone, and satisfies the* (S_+)*-property, that is, any sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that $u_n\rightharpoonup u$ weakly in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle\le 0$ converges strongly to u in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.*

A function $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is said to be a weak solution of (1.8) if

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x\\
&\quad+\int_\Omega \mu(x)\left[\log(e+|\nabla u|)+\frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x\\
&=\lambda\int_\Omega f(x,u)v\,\mathrm{d}x
\end{aligned}$$

is satisfied for all $v\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Furthermore, we define the functionals $\Phi,\Psi,I_\lambda\colon W_0^{1,\mathcal{H}_{\log}}(\Omega)\to\mathbb{R}$ by

$$\begin{aligned}
\Phi(u)&=\int_\Omega\left(\frac{|\nabla u|^{p(x)}}{p(x)}+\mu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\log(e+|\nabla u|)\right)\mathrm{d}x,\\
\Psi(u)&=\int_\Omega F(x,u)\,\mathrm{d}x,\quad\text{where}\quad F(x,t)=\int_0^t f(x,\xi)\,\mathrm{d}\xi\\
I_\lambda(u)&=\Phi(u)-\lambda\Psi(u),
\end{aligned}\tag{2.3}$$

where I_λ is the energy functional associated with our problem (1.8).

Remark 2.6. *Note that, under hypotheses* (H) *and* $(\mathrm{H_f})$, *the functional I_λ is unbounded from below. In order to see this, choose a fixed test function $\varphi\in C_0^\infty(\Omega)$ with $\varphi\ge 0$ and $\varphi\not\equiv 0$. Then $\varphi\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and $\nabla\varphi\in L^\infty(\Omega)$. Hence*

$$|\nabla\varphi(x)|\le C_1\quad\textit{for a.a. } x\in\Omega,$$

for some $C_1>1$. Let $t>0$ sufficiently large. Since $q(x)\le q_+$ and

$$\log(e+t|\nabla\varphi|)\le\log(e+tC_1)\le C_1\log(e+t),$$

we obtain

$$\Phi(t\varphi)\le\int_\Omega\left(t^{q_+}C_1+\|\mu\|_\infty t^{q_+}C_1^2\log(e+t)\right)\mathrm{d}x\le C_2t^{q_+}\log(e+t)\tag{2.4}$$

for some $C_2>0$. Since $\varphi\ge 0$ and $\varphi\not\equiv 0$, there exist a measurable set $E\subset\Omega$ with Lebesgue measure $|E|>0$ and a constant $C_3>0$ such that $\varphi(x)\ge C_3$ for all

$x \in E$. Moreover, by assumption (f$_3$), for every $M > 0$ there exists $T_M > 0$ such that

$$F(x,s) \geq M|s|^{q_+}\log(e+|s|) \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \geq T_M.$$

Because of $t\varphi(x) \geq tC_3$ for $x \in E$ and $t > 0$ is sufficiently large, we have $t\varphi(x) \geq T_M$ for all $x \in E$. Therefore,

$$F(x,t\varphi(x)) \geq M(t\varphi(x))^{q_+}\log(e+t\varphi(x)) \geq M(tC_3)^{q_+}\log(e+tC_3), \quad x \in E,$$

which implies, due to $\log(e+tC_3) \geq C_4\log(e+t)$ for t sufficiently large and $C_4 > 0$, that

$$\begin{aligned}\Psi(t\varphi) = \int_\Omega F(x,t\varphi)\,\mathrm{d}x &\geq \int_E F(x,t\varphi)\,\mathrm{d}x \geq M(tC_3)^{q_+}\log(e+tC_3)|E| \\ &\geq M(tC_3)^{q_+}C_4\log(e+t)|E| = C_5Mt^{q_+}\log(e+t)\end{aligned} \tag{2.5}$$

with $C_5 = C_3^{q_+}C_4|E|$. Combining (2.4) and (2.5), we have

$$I_\lambda(t\varphi) = \Phi(t\varphi) - \lambda\Psi(t\varphi) \leq t^{q_+}\log(e+t)(C_2 - \lambda C_5M)$$

for all sufficiently large t. Since $M > 0$ is arbitrary, we may choose M such that $C_2 - \lambda C_5M < 0$, which gives $I_\lambda(t\varphi) \to -\infty$ as $t \to \infty$, i.e. the functional I_λ is unbounded from below.

We know that the functionals in (2.3) are Gâteaux differentiable with derivatives

$$\begin{aligned}\langle\Phi'(u),v\rangle &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x \\ &\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+||\nabla u)}\right]|\nabla u|^{q(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x, \\ \langle\Psi'(u),v\rangle &= \int_\Omega f(x,u)v\,\mathrm{d}x, \\ \langle I_\lambda'(u),v\rangle &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x \\ &\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u\cdot\nabla v\,\mathrm{d}x \\ &\quad - \lambda\int_\Omega f(x,u)v\,\mathrm{d}x\end{aligned}$$

for all $u,v \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Hence, every critical point $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ of I_λ (i.e. $\langle I_\lambda'(u),v\rangle = 0$ for all $v \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$) is a weak solution of (1.8). Therefore, we approach the problem in finding critical points of the associated energy functional which are then weak solutions of (1.8). Our results rely on an abstract critical point theorem developed by Bonanno–D'Aguì [12, Theorem 2.1 and Remark 2.2], which serves as our primary tool. Before proceeding, we recall the definition of the Cerami condition.

Definition 2.7. *Let $(X,\|\cdot\|)$ be a Banach space, X^* its dual space and $L \in C^1(X)$. We say that the functional L satisfies the Cerami condition,* (C)*-condition for short, if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that*

(C$_1$) *$\{L(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded,*
(C$_2$) *$(1+\|u_n\|)L'(u_n) \to 0$ in X^* as $n \to \infty$,*

has a strongly convergent subsequence in X.

Theorem 2.8. *Let X be a real Banach space and let $\Phi, \Psi \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_{u \in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Assume that Φ is coercive and that there exist $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup\limits_{u \in \Phi^{-1}((-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.6}$$

and for all $\lambda \in \Lambda_r = \left(\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}((-\infty, r])} \Psi(u)} \right)$, the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies the (C)*-condition and it is unbounded from below. Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ admits at least two nontrivial critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.*

3. Proofs of our main results

In this section we present the proofs of Theorems 1.1 and 1.2. Our purpose is to apply Theorem 2.8 to the functionals Φ and Ψ defined in (2.3). We start with the following result.

Proposition 3.1. *Let hypotheses* (H) *and* ($\mathrm{H_f}$) *be satisfied. Then, the functional I_λ satisfies the* (C)*-condition for all $\lambda > 0$.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\mathcal{H}_{\log}}(\Omega)$ be a sequence such that (C_1) and (C_2) hold. The proof is divided into three steps.

Step 1. $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^{\ell_-}(\Omega)$, where $\ell \in C_+(\overline{\Omega})$ is given in (f_4).

From (C_1) one has that there exists a constant $C_1 > 0$ such that $|I_\lambda(u_n)| \leq C_1$ for all $n \in \mathbb{N}$, i.e.,

$$\left| \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x) \frac{|\nabla u|^{q(x)}}{q(x)} \log(e + |\nabla u|) \right) \mathrm{d}x - \lambda \int_\Omega F(x, u_n) \, \mathrm{d}x \right| \leq C_1,$$

which leads to

$$\varrho_{\mathcal{H}_{\log}}(\nabla u_n) - \lambda \int_\Omega q_+ F(x, u_n) \, \mathrm{d}x \leq C_2 \tag{3.1}$$

for some $C_2 > 0$ and for all $n \in \mathbb{N}$. Besides, from (C_2) we know that there exists a sequence $\varepsilon_n \to 0$ such that for all $v \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$

$$\begin{aligned} &\Bigg| \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla v \, \mathrm{d}x \\ &+ \int_\Omega \mu(x) \left[\log(e + |\nabla u_n|) + \frac{\nabla u_n}{q(x)(e + \nabla u_n)} \right] |\nabla u_n|^{q(x)-2} \nabla u_n \cdot \nabla v \, \mathrm{d}x \\ &- \lambda \int_\Omega f(x, u_n) \, \mathrm{d}x \Bigg| \leq \frac{\varepsilon_n \|v\|}{1 + \|u_n\|} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.2}$$

Choosing $v = u_n$ in (3.2) and taking Lemma 2.3 into account one has

$$-\left(1 + \frac{\kappa}{q_-}\right) \varrho_{\mathcal{H}_{\log}}(\nabla u_n) + \lambda \int_\Omega f(x, u_n) u_n \, \mathrm{d}x < \varepsilon_n.$$

On the other hand, multiplying inequality (3.1) by $\left(1+\frac{\kappa}{q_-}\right)>0$, it follows that

$$\left(1+\frac{\kappa}{q_-}\right)\varrho_{\mathcal{H}_{\log}}(\nabla u_n)-\lambda\int_\Omega q_+\left(1+\frac{\kappa}{q_-}\right)F(x,u_n)\,\mathrm{d}x\le C_3$$

for some $C_3>0$ and for all $n\in\mathbb{N}$. Adding both inequalities, one gets

$$\int_\Omega f(x,u_n)u_n\mathrm{d}x-q_+\left(1+\frac{\kappa}{q_-}\right)\int_\Omega F(x,u_n)\,\mathrm{d}x\le C_4$$

for all $n\in\mathbb{N}$ and for some $C_4>0$. Assuming without any loss of generality that $\ell_-\le\tilde{\ell}_-$, from (f$_2$) and (f$_4$), we can find $C_5,C_6>0$

$$f(x,t)t-q_+\left(1+\frac{\kappa}{q_-}\right)F(x,t)\ge C_5|t|^{\ell_-}-C_6.$$

Combining the last two inequalities, it holds that

$$\|u_n\|_{\ell_-}^{\ell_-}\le C_7 \tag{3.3}$$

for some $C_7>0$ and for all $n\in\mathbb{N}$. Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{\ell_-}(\Omega)$.

Step 2. $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.

Note that, from (f$_1$) and (f$_4$), we have that

$$\ell_-<s_+<(p_-)^*.$$

So there exists $t\in(0,1)$ such that

$$\frac{1}{s_+}=\frac{t}{(p_-)^*}+\frac{1-t}{\ell_-}. \tag{3.4}$$

Then, using the interpolation inequality (see Papageorgiou–Winkert [43, Proposition 2.3.17]), one has

$$\|u_n\|_{s_+}\le\|u_n\|_{(p_-)^*}^t\|u_n\|_{\ell_-}^{1-t}.$$

From (3.3), we obtain

$$\|u_n\|_{s_+}\le C_8\|u_n\|_{(p_-)^*}^t$$

for some $C_8>0$ and for all $n\in\mathbb{N}$. Now, choosing $v=u_n$ in (3.2) we have

$$\varrho_{\mathcal{H}_{\log}}(\nabla u_n)-\lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x<\varepsilon_n. \tag{3.5}$$

For simplicity, we can suppose that $\|u_n\|\ge1$ for all $n\in\mathbb{N}$. Taking into account (f$_2$), Proposition 2.2 (iii), by (3.5) it follows that

$$\begin{aligned}\|u_n\|^{p_-}&\le\varrho_{\mathcal{H}_{\log}}(\nabla u_n)<\varepsilon_n+\lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x\\&\le\lambda C(\|u_n\|_1+\|u_n\|_{s_+}^{s_+})+\varepsilon_n.\end{aligned}$$

From the continuous embeddings $L^{s_+}(\Omega)\hookrightarrow L^1(\Omega)$ and $W_0^{1,\mathcal{H}_{\log}}(\Omega)\hookrightarrow L^{s_+}(\Omega)$ (see Proposition 2.4(ii)) we obtain

$$\|u_n\|^{p_-}\le C_9(1+\|u_n\|^{ts_+})+\varepsilon_n \tag{3.6}$$

for some $C_9>0$ and for all $n\in\mathbb{N}$. From (3.4) and hypothesis (f$_4$), we know that

$$ts_+=\frac{(p_-)^*(s_+-\ell_-)}{(p_-)^*-\ell_-}=\frac{Np_-(s_+-\ell_-)}{Np_--N\ell_-+p_-\ell_-}$$

$$< \frac{Np_-(s_+ - \ell_-)}{Np_- - N\ell_- + p_-(s_+ - p_-)\frac{N}{p_-}} = p_-.$$

Using this, the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ follows from (3.6).

Step 3. $u_n \to u$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ up to a subsequence.

Since $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is bounded and $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is a reflexive Banach space, there exists a subsequence (still denoted by u_n) such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{s_+}(\Omega).$$

By exploiting this in (3.2), taking $v = u_n - u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ one has that

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0.$$

Since $A\colon W_0^{1,\mathcal{H}_{\log}}(\Omega) \to W_0^{1,\mathcal{H}_{\log}}(\Omega)^*$ fulfills the (S_+)-property (see Theorem 2.5), we conclude that $u_n \to u$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and the proof is complete. □

Now, we are able to give the proof of Theorem 1.1. First, put

$$R := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega).$$

Then we can find $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$, where $B(x_0, R)$ denotes the ball with center x_0 and radius $R > 0$. We also denote by

$$\omega_R := \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} R^N$$

the Lebesgue measure of the N-dimensional ball of radius R, where

$$\Gamma(t) = \int_0^{+\infty} z^{t-1} e^{-z}\,\mathrm{d}z \quad \text{for all } t > 0$$

is the Gamma function. Next, we put

$$\delta = \frac{\min\{R^{p_-}, R^{q_+}\} p_-}{2^{q_+ + 1 - N} \omega_R (2^N - 1) \max\{1, \|\mu\|_\infty\}}$$

and for any $r, \eta \in \mathbb{R}^+$, we define

$$\alpha(r) = \frac{C\left(k_1 \max\{(q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+ + \kappa}}, \} + \bar{k}_s \max\{(q_+ r)^{\frac{s_+}{p_-}}, (q_+ r)^{\frac{s_-}{q_+ + \kappa}}\}\right)}{r}, \tag{3.7}$$

$$\beta(\eta) = \frac{\delta \displaystyle\int_{B(x_0, \frac{R}{2})} F(x, \eta)\,\mathrm{d}x}{\max\left\{\eta^{p_-}, \eta^{q_+} \log\left(e + \dfrac{2\eta}{R}\right)\right\}}, \tag{3.8}$$

where $\bar{k}_s = \max\{k_s^{s_-}, k_s^{s_+}\}$ and k_1, k_s, C and s are defined in (2.1) and (f_2), respectively.

Proof of Theorem 1.1. Our goal is to apply Theorem 2.8 with $X = W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and Φ as well as Ψ defined as in (2.3). Observe that from Proposition 2.2 (iii) we know that Φ is coercive and from (f_3) it is clear that I_λ is unbounded from below, see Remark 2.6. So, we fix $\lambda \in \Lambda$ (which is nonempty due to (h_2)) and consider a function $\tilde{u} \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ defined as

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,R), \\ \dfrac{2\eta}{R}(R - |x-x_0|) & \text{if } x \in B(x_0,R) \setminus B\left(x_0, \frac{R}{2}\right), \\ \eta & \text{if } x \in B\left(x_0, \frac{R}{2}\right). \end{cases}$$

Taking (1.10) into account, it follows that

$$\begin{aligned} \Phi(\tilde{u}) &= \int_{B(x_0,R)\setminus B(x_0,\frac{R}{2})} \left(\frac{1}{p(x)} \left(\frac{2\eta}{R}\right)^{p(x)} + \frac{\mu(x)}{q(x)} \left(\frac{2\eta}{R}\right)^{q(x)} \log\left(e + \frac{2\eta}{R}\right)\right) \mathrm{d}x \\ &\leq \frac{2^{q_+}}{p_-} \int_{B(x_0,R)\setminus B(x_0,\frac{R}{2})} \left(\left(\frac{\eta}{R}\right)^{p(x)} + \mu(x) \left(\frac{\eta}{R}\right)^{q(x)} \log\left(e + \frac{2\eta}{R}\right)\right) \mathrm{d}x \\ &\leq \frac{2^{q_+}}{p_-} \max\{1, \|\mu\|_\infty\} \max\left\{ \left(\frac{\eta}{R}\right)^{p_-}, \left(\frac{\eta}{R}\right)^{q_+} \log\left(e + \frac{2\eta}{R}\right) \right\} \cdot 2 \cdot (\omega_R - \omega_{\frac{R}{2}}) \\ &= \frac{2^{q_+ + 1 - N}}{p_-} (2^N - 1) \max\{1, \|\mu\|_\infty\} \frac{\max\left\{ \eta^{p_-}, \eta^{q_+} \log\left(e + \frac{2\eta}{R}\right)\right\}}{\min\left\{R^{p_-}, R^{q_+}\right\}} \cdot \omega_R \\ &= \frac{1}{\delta} \max\left\{ \eta^{p_-}, \eta^{q_+} \log\left(e + \frac{2\eta}{R}\right)\right\} < r. \end{aligned}$$

This shows that $0 < \Phi(\tilde{u}) < r$. Now, we prove (2.6). From (h_1) we have

$$\begin{aligned} \Psi(\tilde{u}) &= \int_{B(x_0,R)\setminus B(x_0,\frac{R}{2})} F\left(x, \frac{2\eta}{R}(R - |x-x_0|)\right) \mathrm{d}x + \int_{B(x_0,\frac{R}{2})} F(x,\eta)\,\mathrm{d}x \\ &\geq \int_{B(x_0,\frac{R}{2})} F(x,\eta)\,\mathrm{d}x. \end{aligned}$$

Hence,

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{\delta \displaystyle\int_{B(x_0,\frac{R}{2})} F(x,\eta)\,\mathrm{d}x}{\max\left\{ \eta^{p_-}, \eta^{q_+} \log\left(e + \frac{2\eta}{R}\right)\right\}}. \tag{3.9}$$

Moreover, for $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ satisfying $\Phi(u) \leq r$, one has that

$$q_+ r > q_+ \Phi(u) > \varrho_{\mathcal{H}_{\log}}(\nabla u) \geq \min\left\{ \|u\|^{p_-}, \|u\|^{q_+ + \kappa}\right\}.$$

This implies that

$$\Phi^{-1}((-\infty, r]) \subseteq \left\{ u \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \colon \|u\| < \max\left\{ (q_+ r)^{\frac{1}{p_-}}, (q_+ r)^{\frac{1}{q_+ + \kappa}} \right\}\right\}.$$

From this, (f_2), Proposition 2.1 (iii), (iv) and (2.1) we conclude that

$$\sup_{u \in \Phi^{-1}((-\infty, r])} \Psi(u)$$

$$
\begin{aligned}
&= \sup_{u\in\Phi^{-1}((-\infty,r])} \int_\Omega F(x,u)\,\mathrm{d}x \\
&\leq \sup_{u\in\Phi^{-1}((-\infty,r])} C\int_\Omega \left(|u|+|u|^{s(x)}\right)\,\mathrm{d}x \\
&= \sup_{u\in\Phi^{-1}((-\infty,r])} C\left(\|u\|_1+\varrho_{s(\cdot)}(u)\right) \\
&\leq \sup_{u\in\Phi^{-1}((-\infty,r])} C\left(k_1\|u\|+\max\{\|u\|_{s(\cdot)}^{s_-},\|u\|_{s(\cdot)}^{s_+}\}\right) \\
&\leq \sup_{u\in\Phi^{-1}((-\infty,r])} C\left(k_1\|u\|+\max\{k_s^{s_-},k_s^{s_+}\}\max\{\|u\|^{s_-},\|u\|^{s_+}\}\right) \\
&\leq C\left(k_1\max\big\{(q_+r)^{\frac{1}{p_-}},(q_+r)^{\frac{1}{q_++\kappa}},\big\}+\bar{k}_s\max\{(q_+r)^{\frac{s_+}{p_-}},(q_+r)^{\frac{s_-}{q_++\kappa}}\}\right).
\end{aligned}
$$

Now, taking (h_2) and (3.9) into account, one has

$$
\begin{aligned}
&\frac{\sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{r} \\
&\leq \frac{C\left(k_1\max\big\{(q_+r)^{\frac{1}{p_-}},(q_+r)^{\frac{1}{q_++\kappa}},\big\}+\bar{k}_s\max\{(q_+r)^{\frac{s_+}{p_-}},(q_+r)^{\frac{s_-}{q_++\kappa}}\}\right)}{r} \\
&\leq \beta(\eta)\leq \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.
\end{aligned}
$$

Thus, condition (2.6) is verified and from Proposition 3.1 we know that I_λ fulfills the (C)-condition. Therefore, we can apply Theorem 2.8 and obtain two nontrivial weak solutions $u_{\lambda,1},u_{\lambda,2}\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ of (1.8) such that $I_\lambda(u_{\lambda,1})<0<I_\lambda(u_{\lambda,2})$. From Theorem 3.1 by Rădulescu–Stapenhorst–Winkert [45] we know that these solutions are bounded as well. □

A direct consequence about nonnegative solutions is the following corollary.

Corollary 3.2. *Suppose that, in addition to the assumptions of Theorem 1.1, $f(x,0)\geq 0$ and $f(x,t)=f(x,0)$ for a.a. $x\in\Omega$ and for all $t<0$. Then, problem (1.8) admits at least two nontrivial and nonnegative bounded weak solutions with opposite energy sign.*

Proof. Applying Theorem 1.1 gives us two bounded nontrivial weak solutions $u_{\lambda,1}$ and $u_{\lambda,2}$ of (1.8). We only have to prove the nonnegativity. Testing the weak formulation of problem (1.8) related to $u_{\lambda,1}$ with $v=-u_{\lambda,1}^-=-\max\{-u_{\lambda,1},0\}\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ (see [5, Proposition 3.8 (iii)]) we obtain

$$
\begin{aligned}
&\int_\Omega \Big(|\nabla u_{\lambda,1}|^{p(x)-2}\nabla u_{\lambda,1}\cdot\nabla(-u_{\lambda,1}^-)\,\mathrm{d}x \\
&+\int_\Omega \mu(x)\left[\log(e+|\nabla u|)+\frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u_{\lambda,1}|^{q(x)-2}\nabla u_{\lambda,1}\cdot\nabla(-u_{\lambda,1}^-)\,\mathrm{d}x \\
&=\lambda\int_\Omega f(x,u_{\lambda,1})\,u_{\lambda,1}^-\,\mathrm{d}x,
\end{aligned}
$$

and so

$$
-\varrho_{\mathcal{H}_{\log}}(\nabla u_{\lambda,1}^-)\geq 0.
$$

On the other hand, Proposition 2.2 (iii) gives

$$\min\left\{\|u_{\lambda,1}\|^{p_-},\|u_{\lambda,1}\|^{q_++\kappa}\right\}\le \varrho_{\mathcal{H}_{\log}}(u_{\lambda,1})\le 0,$$

which implies that $\|u_{\lambda,1}^-\|=0$. Then, $u_{\lambda,1}^-=0$ and $u_{\lambda,1}\ge 0$. The same argument shows that $u_{\lambda,2}\ge 0$. □

Finally, we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. From condition (1.11) we have

$$\begin{aligned}\limsup_{\eta\to 0^+}\beta(\eta)&=\limsup_{\eta\to 0^+}\delta\frac{\int_{B(x_0,\frac{R}{2})}F(x,\eta)\,\mathrm{d}x}{\max\left\{\eta^{p_-},\eta^{q_+}\log\left(e+\frac{2\eta}{R}\right)\right\}}\\&\ge\delta\omega_{\frac{R}{2}}\limsup_{\eta\to 0^+}\frac{\inf_{x\in\Omega}F(x,\eta)}{\eta^{p_-}}=+\infty.\end{aligned}\tag{3.10}$$

Thus, fixing $\lambda\in]0,\lambda^*)$, we can choose $r>0$ such that

$$\begin{aligned}\lambda&<\frac{1}{\alpha(r)}\\&=\frac{r}{C\left(k_1\max\left\{(q_+r)^{\frac{1}{p_-}},(q_+r)^{\frac{1}{q_++\kappa}}\right\}+\bar{k}_s\max\left\{(q_+r)^{\frac{s_+}{p_-}},(q_+r)^{\frac{s_-}{q_++\kappa}}\right\}\right)}.\end{aligned}$$

Next, from (3.10), we deduce that there exists $\eta>0$ small enough such that

$$\delta\omega_{\frac{R}{2}}\frac{\inf_{x\in\Omega}F(x,\eta)}{\eta^{p_-}}>\frac{1}{\lambda}.$$

This implies that $\alpha(r)<\beta(\eta)$. Finally, applying Theorem 1.1 and following the arguments used in the proof of Corollary 3.2, we conclude that problem (1.8) admits at least two nontrivial, nonnegative, bounded weak solutions with opposite energy signs, as required. □

Finally, we provide an example of a function $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfying the assumptions ($\mathrm{H_f}$).

Example 3.3. *Let $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ be defined by*

$$f(x,t)=\begin{cases}|t|^{\alpha(x)-2}t, & |t|<1,\\ |t|^{\beta(x)-2}t\left(\log|t|+1\right), & |t|\ge 1,\end{cases}$$

where $\alpha,\beta\in C(\overline{\Omega})$ such that

$$q_+<\beta(x)<(p_-)^*\quad\text{for all } x\in\overline{\Omega}\quad\text{and}\quad\frac{\beta_+}{p_-}-\frac{\beta_-}{N}<1.$$

By construction, f satisfies the Carathéodory condition ($\mathrm{f_1}$). Moreover, setting $l(x)=\beta(x)$ for all $x\in\overline{\Omega}$ and $s(x)=\beta(x)+\sigma$ for all $x\in\overline{\Omega}$ for some sufficiently small $\sigma>0$ such that

$$\frac{s_+}{p_-}-\frac{\beta_-}{N}<1\quad\text{and}\quad s_+<(p_-)^*,$$

conditions ($\mathrm{f_2}$), ($\mathrm{f_3}$), and ($\mathrm{f_4}$) are satisfied.

If, in addition, $\alpha(x)<p_-$ for all $x\in\overline{\Omega}$, then Theorem 1.2 applies to

$$f_+(x,t)=|f(x,t)|$$

for every $(x,t)\in\Omega\times\mathbb{R}$.

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