

# $L^\infty$ -ESTIMATES FOR NONLINEAR ELLIPTIC NEUMANN BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper we prove the  $L^\infty$ -boundedness of solutions of the quasilinear elliptic equation

$$\begin{aligned} Au &= f(x, u, \nabla u) && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x, u) && \text{on } \partial\Omega, \end{aligned}$$

where  $A$  is a second order quasilinear differential operator and  $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  as well as  $g : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are Carathéodory functions satisfying natural growth conditions. Our main result is given in Theorem 4.1 and is based on the Moser iteration technique along with real interpolation theory. Furthermore, the solutions of the elliptic equation above belong to  $C^{1,\alpha}(\bar{\Omega})$ , if  $g$  is Hölder continuous.

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with Lipschitz boundary  $\partial\Omega$ . We consider the quasilinear elliptic equation

$$\begin{aligned} Au &= f(x, u, \nabla u) && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x, u) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where  $\frac{\partial u}{\partial \nu}$  denotes the conormal derivative of  $u$ . Here,  $A$  is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)),$$

and  $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  as well as  $g : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are some Carathéodory functions. For  $u \in W^{1,p}(\Omega)$  defined on the boundary  $\partial\Omega$ , we make use of the trace operator  $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$  which is well known to be compact. For easy readability we will drop the notation  $\gamma u$  and write for short  $u$ , respectively,  $g(x, u) := g(x, \gamma u)$ . The main goal of this paper is to prove a priori estimates for solutions of the nonlinear elliptic equation in (1.1). For this purpose, we use some important tools like the Moser iteration technique and real interpolation theory. By an a priori estimate, we mean an estimate of the form

$$\|u\|_{L^\infty(\Omega)} \leq C,$$

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for all possible solutions of (1.1) with some constant  $C$  independent of  $u$ . Concerning a priori bounds for elliptic equations with zero Neumann conditions we refer to the results in [17] and [19], where they consider problems of the form

$$\begin{aligned} -\Delta u + \lambda u &= f(u), \quad u > 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

in a bounded convex domain  $\Omega \subset \mathbb{R}^3$  with smooth boundary and  $\lambda > 0$ . Motreanu et al. have applied the Moser iteration, too, in [11, Proof of Proposition 2.5] to prove  $L^\infty$ -boundedness for solutions of the Neumann problem

$$\begin{aligned} -\operatorname{div} \vartheta_\varepsilon(z, \nabla v_\varepsilon) &= f_0(z, v_\varepsilon) + \lambda_\varepsilon f_0(z, x_0) - \lambda_\varepsilon |v_\varepsilon - x_0|^{p-2} (v_\varepsilon - x_0) && \text{in } Z, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial Z, \end{aligned}$$

where

$$\vartheta_\varepsilon(z, \xi) = |\xi|^{p-2} \xi + \lambda_\varepsilon |\nabla x_0|^{p-2} \nabla x_0 + \lambda_\varepsilon |\xi - \nabla x_0|^{p-2} (\xi - \nabla x_0),$$

with  $Z \subset \mathbb{R}^N$  is a bounded domain with a  $C^2$ -boundary  $\partial Z$ ,  $0 < \lambda_\varepsilon \leq 1$ ,  $\varepsilon \in (0, 1]$ ,  $x_0 \in L^\infty(\Omega)$  fixed and with a Carathéodory function  $f_0 : Z \times \mathbb{R} \rightarrow \mathbb{R}$  satisfying some growth condition.

The novelty of our paper is the demonstration of a priori estimates for nonlinear elliptic equations with nonlinear nonhomogenous Neumann boundary values of the form (1.1), where the Carathéodory functions  $f$  and  $g$  depend on  $u$ ,  $\nabla u$  and  $u$ , respectively, satisfying a natural growth condition. Additionally, we extend our results and show that every solution of (1.1) belongs to  $C^{1,\alpha}(\bar{\Omega})$  in case  $g$  satisfies the condition

$$|g(x_1, s_1) - g(x_2, s_2)| \leq L \left[ |x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha \right],$$

for all pairs  $(x_1, s_1), (x_2, s_2)$  in  $\partial\Omega \times [-M_0, M_0]$ , where  $M_0$  is a positive constant and  $\alpha \in (0, 1]$ . The  $C^1$ -regularity follows directly from the  $L^\infty$ -boundedness along with the results of Liebermann in [10].

## 2. NOTATIONS AND HYPOTHESES

Let  $\frac{1}{p} + \frac{1}{q} = 1$ . We suppose the following conditions on the operator  $A$  and the nonlinearities  $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  and  $g : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ .

- (A1) Each  $a_i(x, s, \xi)$  satisfies Carathéodory conditions, i.e., is measurable in  $x \in \Omega$  for all  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$  and continuous in  $(s, \xi)$  for a.e.  $x \in \Omega$ . Furthermore, a constant  $c_0 > 0$  and a function  $k_0 \in L^q(\Omega)$  exist so that

$$|a_i(x, s, \xi)| \leq k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1}),$$

for a.e.  $x \in \Omega$  and for all  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ , where  $|\xi|$  denotes the Euclidian norm of the vector  $\xi$ .

- (A2) The coefficients  $a_i$  satisfy a monotonicity condition with respect to  $\xi$  in the form

$$\sum_{i=1}^N (a_i(x, s, \xi) - a_i(x, s, \xi')) (\xi_i - \xi'_i) > 0,$$

for a.e.  $x \in \Omega$ , for all  $s \in \mathbb{R}$ , and for all  $\xi, \xi' \in \mathbb{R}^N$  with  $\xi \neq \xi'$ .

(A3) A constant  $c_1 > 0$  and a function  $k_1 \in L^\infty(\Omega)$  exist such that

$$\sum_{i=1}^N a_i(x, s, \xi) \xi_i \geq c_1 |\xi|^p - k_1(x),$$

for a.e.  $x \in \Omega$ , for all  $s \in \mathbb{R}$ , and for all  $\xi \in \mathbb{R}^N$ .

(F1)  $x \mapsto f(x, s, \xi)$  is measurable in  $\Omega$  for all  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ .

(F2)  $(s, \xi) \mapsto f(x, s, \xi)$  is continuous in  $\mathbb{R} \times \mathbb{R}^N$  for almost all  $x \in \Omega$ .

(F3) There exists a constant  $c_2 > 0$  such that

$$|f(x, s, \xi)| \leq c_2(1 + |s|^{p-1} + |\xi|^{p-1}), \quad (2.1)$$

for a.e.  $x \in \Omega$ , for all  $s \in \mathbb{R}$  and for all  $\xi \in \mathbb{R}^N$ .

(G1)  $x \mapsto g(x, s)$  is measurable in  $\partial\Omega$  for all  $s \in \mathbb{R}$ .

(G2)  $s \mapsto g(x, s)$  is continuous in  $\mathbb{R}$  for almost all  $x \in \partial\Omega$ .

(G3) There exists a constant  $c_3 > 0$  such that

$$|g(x, s)| \leq c_3(1 + |s|^{p-1}), \quad (2.2)$$

for a.e.  $x \in \partial\Omega$  and for all  $s \in \mathbb{R}$ .

Condition (A1) implies that  $A : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*$  is bounded continuous and along with (A2) it holds that  $A$  is pseudomonotone. Due to (A1) the operator  $A$  generates a mapping from  $W^{1,p}(\Omega)$  into its dual space defined by

$$\langle Au, \varphi \rangle = \int_{\Omega} \sum_{i=1}^N a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} dx,$$

where  $\langle \cdot, \cdot \rangle$  stands for the duality pairing between  $W^{1,p}(\Omega)$  and  $(W^{1,p}(\Omega))^*$ . Assumption (A3) is a coercivity type condition. The conditions (F3) and (G3) ensure that the corresponding Nemytskij operators  $F : W^{1,p}(\Omega) \rightarrow L^q(\Omega)$  and  $G : L^p(\partial\Omega) \rightarrow L^q(\partial\Omega)$  defined by

$$Fu(x) = f(x, u(x), \nabla u(x)), \quad Gu(x) = g(x, u(x)),$$

are bounded and continuous (see e.g. [18]). The definition of a solution of problem (1.1) in the weak sense is defined as follows.

**Definition 2.1.** *A function  $u \in W^{1,p}(\Omega)$  is said to be a weak solution of (1.1) if the following holds*

$$\int_{\Omega} \sum_{i=1}^N a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} dx = \int_{\Omega} f(x, u, \nabla u) \varphi dx + \int_{\partial\Omega} g(x, u) \varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega).$$

**Remark 2.2.** *The growth conditions on the function  $f$  and  $g$  can be relaxed, replacing  $|s|^{p-1}$  by  $|s|^q$  for a suitable  $q > p-1$ . Thanks to the Sobolev embedding and to the trace embedding, the definition of a weak solution to the Neumann problem would also be consistent in this case. For reasons of simplification, we deal with the given growth conditions as in (2.1) and (2.2).*

### 3. THE SPACES $B_{pq}^s$ AND $F_{pq}^s$

In this section, we give a brief overview about Besov spaces (respectively, Lizorkin-Triebel spaces) which are needed in the proof of our main theorem.

If  $\mathcal{A}$  is a Banach space, then

$$l_p^\sigma(\mathcal{A}) = \left\{ \xi : \xi = \{\xi_j\}_{j=0}^\infty; \xi_j \in \mathcal{A}; \|\xi\|_{l_p^\sigma} = \left( \sum_{j=0}^\infty 2^{j\sigma p} \|\xi_j\|_{\mathcal{A}}^p \right)^{\frac{1}{p}} < \infty \right\}$$

for  $1 \leq p < \infty$  and

$$l_\infty^\sigma(\mathcal{A}) = \left\{ \xi : \xi = \{\xi_j\}_{j=0}^\infty; \xi_j \in \mathcal{A}; \|\xi\|_{l_\infty^\sigma} = \sup_j 2^{j\sigma} \|\xi_j\|_{\mathcal{A}} < \infty \right\}$$

for  $p = \infty$  are also Banach spaces (cf. [14, Section 1.18]) where  $\sigma$  is a real number. We recall that  $S = S(\mathbb{R}^N)$  is the set of all complex-valued rapidly decreasing infinitely differentiable functions defined on the  $N$ -dimensional real Euclidean space  $\mathbb{R}^N$ . The spaces  $S(\mathbb{R}^N)$  and  $S'(\mathbb{R}^N)$  (dual space) have their usual topologies, where  $S'(\mathbb{R}^N)$  is equipped with the strong topology. We denote by  $\mathcal{F}$  the Fourier transform in  $S$  and the support of a distribution  $f$  is written as  $\text{supp } f$ . Further, we set

$$M_j = \{\xi : \xi \in \mathbb{R}^N, 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \quad j = 1, 2, \dots,$$

$$M_0 = \{\xi : \xi \in \mathbb{R}^N, |\xi| \leq 2\}.$$

Then we introduce the spaces  $B_{pq}^s(\mathbb{R}^N)$  and  $F_{pq}^s(\mathbb{R}^N)$  as follows (see [14, Definition 2.3.1/1]).

**Definition 3.1.**

(a) For  $-\infty < s < \infty$ ,  $1 < p < \infty$ , and  $1 \leq q < \infty$  one sets

$$B_{pq}^s(\mathbb{R}^N) = \left\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^\infty a_j(x); \text{supp } \mathcal{F}a_j \subset M_j; \right. \\ \left. \|\{a_j\}\|_{l_q^s(L_p)} = \left[ \sum_{j=0}^\infty (2^{sj} \|a_j(x)\|_{L_p})^q \right]^{\frac{1}{q}} < \infty \right\}$$

and for  $-\infty < s < \infty$ ,  $1 < p < \infty$ , and  $q = \infty$  one sets

$$B_{p\infty}^s(\mathbb{R}^N) = \left\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^\infty a_j(x); \text{supp } \mathcal{F}a_j \subset M_j; \right. \\ \left. \|\{a_j\}\|_{l_\infty^s(L_p)} = \sup_j 2^{sj} \|a_j(x)\|_{L_p} < \infty \right\}.$$

Further, for  $-\infty < s < \infty$ ,  $1 < p < \infty$  and  $1 \leq q \leq \infty$  let

$$\|f\|_{B_{pq}^s(\mathbb{R}^N)} = \inf_{f=\sum a_j} \|\{a_j\}\|_{l_q^s(L_p)}.$$

(b) For  $-\infty < s < \infty$ ,  $1 < p < \infty$ , and  $1 < q < \infty$  one sets

$$F_{pq}^s(\mathbb{R}^N) = \left\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^{\infty} a_j(x); \text{supp } \mathcal{F}a_j \subset M_j; \right. \\ \left. \|\{a_j\}\|_{L_p(l_q^s)} = \left[ \int_{\Omega} \left( \sum_{j=0}^{\infty} 2^{sjq} |a_j(x)|^q \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} < \infty \right\}.$$

Further, let

$$\|f\|_{F_{pq}^s(\mathbb{R}^N)} = \inf_{f=\sum a_j} \|\{a_j\}\|_{L_p(l_q^s)}.$$

(c) For  $-\infty < s < \infty$  and  $1 < p < \infty$  one sets

$$H_p^s(\mathbb{R}^N) = F_{p2}^s(\mathbb{R}^N).$$

(d) For  $1 < p < \infty$  one sets

$$W_p^s(\mathbb{R}^N) = \begin{cases} H_p^s(\mathbb{R}^N) & \text{if } s = 0, 1, 2, \dots, \\ B_{pp}^s(\mathbb{R}^N) & \text{if } 0 < s \neq \text{integer}. \end{cases}$$

The proof of the following theorem can be found in [14, Theorem 2.3.2].

**Theorem 3.2.**

- (1) Let  $-\infty < s < \infty$ ,  $1 < p < \infty$ , and  $1 \leq q \leq \infty$ . Then  $B_{pq}^s(\mathbb{R}^N)$  is a Banach space.
- (2) Let  $-\infty < s < \infty$ ,  $1 < p < \infty$ , and  $1 < q < \infty$ . Then  $F_{pq}^s(\mathbb{R}^N)$  is a Banach space.

It is clear, that all notations above hold true if we replace  $\mathbb{R}^N$  by a bounded domain  $\Omega \subset \mathbb{R}^N$ . The spaces  $B_{pq}^s$  and  $F_{pq}^s$  are called Besov and Lizorkin-Triebel spaces, respectively, which are equal in case  $p = q$  with  $1 < p < \infty$  and  $-\infty < s < \infty$ . We see that the spaces  $W_p^s$  with  $s = 1, 2, 3, \dots$  coincide with the well-known Sobolev spaces introduced by S.L. Sobolev and the extension of the definition of the spaces  $W_p^s$  to values  $0 < s \neq \text{integer}$  are the so-called Slobodeckij spaces. Finally, it was shown that  $H_p^s$  with  $s > 0$  coincide with the well-known Lebesgue (or Liouville, or Bessel-potential) spaces. For more details we refer for example to the books of H. Triebel in [14, 15, 16] or to the monograph of Runst and Sickel in [12].

In our considerations, we need the following continuous embeddings

$$T_1 : B_{pp}^s(\Omega) \rightarrow B_{pp}^{s-\frac{1}{p}}(\partial\Omega), \quad \text{with } s > \frac{1}{p}, \\ T_2 : B_{pp}^{s-\frac{1}{p}}(\partial\Omega) = F_{pp}^{s-\frac{1}{p}}(\partial\Omega) \rightarrow F_{p2}^0(\partial\Omega) = L^p(\partial\Omega), \quad \text{with } s > \frac{1}{p},$$

where  $\Omega$  is a bounded  $C^\infty$ -domain (see [12, Page 75 and Page 82], [14, 2.3.1 and 2.3.2] and [15, 3.3.1]). Let  $s = m + \iota$  with  $m \in \mathbb{N}_0$  and  $0 \leq \iota < 1$ . Then the embeddings are also valid if  $\partial\Omega \in C^{m,1}$  ([13]). In [3, Satz 9.40] the proof is given for  $p = 2$ , however, it can be extended to  $p \in (1, \infty)$  by using the Fourier transformation in  $L^p(\Omega)$  ([4]).

We set  $s = \frac{1}{p} + \tilde{\varepsilon}$ , where  $\tilde{\varepsilon} > 0$  is arbitrarily fixed such that  $s = \frac{1}{p} + \tilde{\varepsilon} < 1$ . Thus, the embeddings are valid for a Lipschitz boundary  $\partial\Omega$ . This yields

$$T_3 : B_{pp}^{\frac{1}{p} + \tilde{\varepsilon}}(\Omega) \rightarrow L^p(\partial\Omega),$$

which means

$$\|v\|_{L^p(\partial\Omega)} \leq c_4 \|v\|_{B_{pp}^{\frac{1}{p} + \tilde{\varepsilon}}(\Omega)}, \quad \forall v \in B_{pp}^{\frac{1}{p} + \tilde{\varepsilon}}(\Omega), \quad (3.1)$$

where  $c_4$  is a positive constant. The real interpolation theory implies

$$(F_{p2}^0(\Omega), F_{p2}^1(\Omega))_{\frac{1}{p} + \tilde{\varepsilon}, p} = (L^p(\Omega), W^{1,p}(\Omega))_{\frac{1}{p} + \tilde{\varepsilon}, p} = B_{pp}^{\frac{1}{p} + \tilde{\varepsilon}}(\Omega),$$

(see [1], [14], [15], [16, Section 1.6.2 and 1.6.7]) which ensures the norm estimate

$$\|v\|_{B_{pp}^{\frac{1}{p} + \tilde{\varepsilon}}(\Omega)} \leq c_5 \|v\|_{W^{1,p}(\Omega)}^{\frac{1}{p} + \tilde{\varepsilon}} \|v\|_{L^p(\Omega)}^{1 - \frac{1}{p} - \tilde{\varepsilon}}, \quad \forall v \in W^{1,p}(\Omega), \quad (3.2)$$

(cf. [14, Theorem 1.3.3 (g)]) with a positive constant  $c_5$  only depending on the boundary  $\partial\Omega$ .

#### 4. MAIN RESULTS

**Theorem 4.1.** *Let the conditions (A1)–(A3), (F1)–(F3) and (G1)–(G3) be satisfied. Let  $u \in W^{1,p}(\Omega)$  be a solution of (1.1). Then  $u \in L^\infty(\Omega)$ .*

*Proof.* To prove the  $L^\infty$ -regularity of  $u$ , we will use the Moser iteration technique (see e.g. [5], [6], [7], [8], [9]). It suffices to consider the proof in case  $1 \leq p \leq N$ , otherwise we would be done. First we are going to show that  $u^+ = \max\{u, 0\}$  belongs to  $L^\infty(\Omega)$ . For  $M > 0$  we define  $v_M(x) = \min\{u^+(x), M\}$ . Letting  $K(t) = t$  if  $t \leq M$  and  $K(t) = M$  if  $t > M$ , it follows by [9, Theorem B.3] that  $K \circ u^+ = v_M \in W^{1,p}(\Omega)$  and hence  $v_M \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ . For real  $k \geq 0$  we choose  $\varphi = v_M^{kp+1}$ , then  $\nabla\varphi = (kp+1)v_M^{kp}\nabla v_M$  and  $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ . Notice that  $u(x) \leq 0$  implies directly  $v_M(x) = 0$ . Testing the weak formulation in Definition 2.1 with  $\varphi = v_M^{kp+1}$ , one gets

$$\begin{aligned} & (kp+1) \int_{\Omega} \sum_{i=1}^N a_i(x, u^+, \nabla u^+) v_M^{kp} \frac{\partial v_M}{\partial x_i} dx \\ &= \int_{\Omega} f(x, u^+, \nabla u^+) v_M^{kp+1} dx + \int_{\partial\Omega} g(x, u^+) v_M^{kp+1} d\sigma. \end{aligned} \quad (4.1)$$

Applying condition (A3) and the Hölder inequality, the left-hand side of (4.1) can be estimated to obtain

$$\begin{aligned}
 & (kp+1) \int_{\Omega} \sum_{i=1}^N a_i(x, u^+, \nabla u^+) v_M^{kp} \frac{\partial v_M}{\partial x_i} dx \\
 &= (kp+1) \int_{\Omega} \sum_{i=1}^N a_i(x, v_M, \nabla v_M) \frac{\partial v_M}{\partial x_i} v_M^{kp} dx \\
 &\geq (kp+1) \int_{\Omega} (c_1 |\nabla v_M|^p - k_1) v_M^{kp} dx \tag{4.2} \\
 &\geq c_1 \frac{kp+1}{(k+1)^p} \int_{\Omega} |\nabla v_M^{k+1}|^p dx - e_1 (kp+1) \int_{\Omega} (u^+)^{kp} dx \\
 &\geq c_1 \frac{kp+1}{(k+1)^p} \int_{\Omega} |\nabla v_M^{k+1}|^p dx - e_1 (kp+1) |\Omega|^{\frac{1}{k+1}} \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp}{(k+1)p}},
 \end{aligned}$$

where  $e_1 = \|k_1\|_\infty$ . The assumption (F3) along with the Hölder inequality and Young's inequality implies

$$\begin{aligned}
 & \int_{\Omega} f(x, u^+, \nabla u^+) v_M^{kp+1} dx \\
 &\leq c_2 \int_{\Omega} (1 + |u^+|^{p-1} + |\nabla u^+|^{p-1}) v_M^{kp+1} dx \\
 &\leq c_2 |\Omega|^{\frac{p-1}{(k+1)p}} \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp+1}{(k+1)p}} + c_2 \int_{\Omega} (u^+)^{(k+1)p} dx \tag{4.3} \\
 &\quad + c_2 \int_{\Omega} \delta |\nabla u^+|^{(p-1)q} v_M^{k(p-1)q} dx + c_2 \int_{\Omega} C(\delta) v_M^{(k+1)p} dx \\
 &\leq e_2 \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp+1}{(k+1)p}} + (1 + C(\delta)) c_2 \int_{\Omega} (u^+)^{(k+1)p} dx \\
 &\quad + \frac{c_2 \delta}{(k+1)^p} \int_{\Omega} |\nabla (u^+)^{k+1}|^p dx.
 \end{aligned}$$

The same arguments for the boundary integral provide

$$\begin{aligned}
 \int_{\partial\Omega} g(x, u^+) v_M^{kp+1} d\sigma &\leq c_3 \int_{\partial\Omega} (1 + |u^+|^{p-1}) v_M^{kp+1} d\sigma \\
 &\leq e_3 \left( \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma \right)^{\frac{kp+1}{(k+1)p}} + e_4 \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma. \tag{4.4}
 \end{aligned}$$

Applying the estimates (4.2)–(4.4) to (4.1) one gets

$$\begin{aligned}
& \frac{kp+1}{(k+1)^p} \int_{\Omega} |\nabla v_M^{k+1}|^p dx \\
& \leq e_2 \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp+1}{(k+1)^p}} + (1+C(\delta))c_2 \int_{\Omega} (u^+)^{(k+1)p} dx \\
& \quad + \frac{c_2\delta}{(k+1)^p} \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx + e_3 \left( \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma \right)^{\frac{kp+1}{(k+1)^p}} \\
& \quad + e_4 \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma + e_5(kp+1) \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp}{(k+1)^p}}.
\end{aligned}$$

We have  $\lim_{M \rightarrow \infty} v_M(x) = u^+(x)$  for a.a.  $x \in \Omega$  and can apply Fatou's Lemma which results in

$$\begin{aligned}
& \frac{kp+1}{(k+1)^p} \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx \\
& \leq e_2 \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp+1}{(k+1)^p}} + (1+C(\delta))c_2 \int_{\Omega} (u^+)^{(k+1)p} dx \\
& \quad + \frac{c_2\delta}{(k+1)^p} \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx + e_3 \left( \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma \right)^{\frac{kp+1}{(k+1)^p}} \\
& \quad + e_4 \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma + e_5(kp+1) \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp}{(k+1)^p}}.
\end{aligned} \tag{4.5}$$

We have either

$$\left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp+1}{(k+1)^p}} \leq 1 \quad \text{or} \quad \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp+1}{(k+1)^p}} \leq \int_{\Omega} (u^+)^{(k+1)p} dx,$$

respectively, either

$$\left( \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma \right)^{\frac{kp+1}{(k+1)^p}} \leq 1 \quad \text{or} \quad \left( \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma \right)^{\frac{kp+1}{(k+1)^p}} \leq \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma,$$

and finally, either

$$\left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp}{(k+1)^p}} \leq 1 \quad \text{or} \quad \left( \int_{\Omega} (u^+)^{(k+1)p} dx \right)^{\frac{kp}{(k+1)^p}} \leq \int_{\Omega} (u^+)^{(k+1)p} dx.$$

Using the calculation above to (4.5), we obtain

$$\begin{aligned}
& \left[ \frac{kp+1}{(k+1)^p} - \frac{c_2\delta}{(k+1)^p} \right] \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx \\
& \leq (C(\delta)c_2 + e_6(kp+1)) \int_{\Omega} (u^+)^{(k+1)p} dx + e_7 \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma + e_8,
\end{aligned}$$

where the choice  $\delta = \frac{kp+1}{2c_2}$  results in

$$\begin{aligned}
& \frac{kp+1}{2(k+1)^p} \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx \\
& \leq e_9(kp+1) \int_{\Omega} (u^+)^{(k+1)p} dx + e_7 \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma + e_8.
\end{aligned} \tag{4.6}$$



It should be pointed out that

$$C(\delta) = (\delta p)^{-\frac{q}{p}} \cdot \frac{1}{q} = \left(\frac{2c_2}{p}\right)^{\frac{q}{p}} \cdot \left(\frac{1}{kp+1}\right)^{\frac{q}{p}} \cdot \frac{1}{q} \leq e_{10}$$

with a positive constant  $e_{10}$  where we have set  $e_9 = e_{10}c_2 + e_6$ . Adding on both sides of (4.6) the positive integral  $\frac{kp+1}{2(k+1)^p} \int_{\Omega} (u^+)^{(k+1)p} dx$  yields

$$\begin{aligned} & \frac{kp+1}{2(k+1)^p} \left[ \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx + \int_{\Omega} (u^+)^{(k+1)p} dx \right] \\ & \leq e_{11}(kp+1) \int_{\Omega} (u^+)^{(k+1)p} dx + e_7 \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma + e_8, \end{aligned} \quad (4.7)$$

due to the fact that  $\frac{kp+1}{2(k+1)^p} < kp+1$  for all  $k \geq 0$ . Next we want to estimate the boundary integral by an integral in the domain  $\Omega$ . Using (3.1), (3.2) and Young's inequality yields

$$\begin{aligned} & \int_{\partial\Omega} ((u^+)^{k+1})^p d\sigma \\ & = \|(u^+)^{k+1}\|_{L^p(\partial\Omega)}^p \\ & \leq c_4^p \|(u^+)^{k+1}\|_{B_{\frac{1}{p}}^{\frac{1}{p}+\varepsilon}(\Omega)}^p \\ & \leq c_4^p c_5^p \|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}^{(\frac{1}{p}+\varepsilon)p} \|(u^+)^{k+1}\|_{L^p(\Omega)}^{(1-\frac{1}{p}-\varepsilon)p} \\ & \leq c_4^p c_5^p \left( \delta' \|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}^{(1+\varepsilon p)\tilde{q}} + C(\delta') \|(u^+)^{k+1}\|_{L^p(\Omega)}^{(p-1-\varepsilon p)\tilde{q}'} \right) \\ & = c_4^p c_5^p \left( \delta' \|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}^p + C(\delta') \|(u^+)^{k+1}\|_{L^p(\Omega)}^p \right), \end{aligned} \quad (4.8)$$

where  $\tilde{q} = \frac{p}{1+\varepsilon p}$  and  $\tilde{q}' = \frac{p}{p-1-\varepsilon p}$  satisfy  $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$  and  $\delta'$  is a free parameter specified later. Note that the positive constant  $C(\delta')$  depends on  $\delta'$ . Applying (4.8) to (4.7) shows

$$\begin{aligned} & \frac{kp+1}{2(k+1)^p} \left[ \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx + \int_{\Omega} (u^+)^{(k+1)p} dx \right] \\ & \leq e_{11}(kp+1) \int_{\Omega} (u^+)^{(k+1)p} dx + e_7 \int_{\partial\Omega} (u^+)^{(k+1)p} d\sigma + e_8 \\ & \leq e_{11}(kp+1) \int_{\Omega} (u^+)^{(k+1)p} dx + e_{12}\delta' \|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}^p \\ & \quad + e_{12}C(\delta') \|(u^+)^{k+1}\|_{L^p(\Omega)}^p + e_8, \end{aligned}$$

where  $e_{12} = e_7 c_4^p c_5^p$  is a positive constant. We take  $\delta' = \frac{kp+1}{e_{12}4(k+1)^p}$  to get

$$\begin{aligned} & \left( \frac{kp+1}{2(k+1)^p} - e_{12} \frac{kp+1}{e_{12}4(k+1)^p} \right) \left[ \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx + \int_{\Omega} (u^+)^{(k+1)p} dx \right] \\ & \leq e_{11}(kp+1) \int_{\Omega} (u^+)^{(k+1)p} dx + e_{12}C(\delta') \|(u^+)^{k+1}\|_{L^p(\Omega)}^p + e_8 \\ & \leq e_{13}(kp+1 + C(\delta')) \int_{\Omega} (u^+)^{(k+1)p} dx + e_8, \end{aligned} \quad (4.9)$$

where it holds

$$\begin{aligned} kp + 1 + C(\delta') &= kp + 1 + \left(\frac{4e_{12}}{p}\right)^{\frac{q}{p}} \cdot \left(\frac{(k+1)^p}{kp+1}\right)^{\frac{q}{p}} \cdot \frac{1}{q} \\ &\leq e_{14} \left( kp + 1 + \left(\frac{(k+1)^{\frac{p}{p-1}}}{(kp+1)^{\frac{1}{p-1}}}\right) \right) \\ &\leq e_{15}(kp+1)^{\frac{p}{p-1}}. \end{aligned}$$

Applying the calculations above to (4.9) provides

$$\begin{aligned} &\frac{kp+1}{4(k+1)^p} \left[ \int_{\Omega} |\nabla(u^+)^{k+1}|^p dx + \int_{\Omega} |(u^+)^{k+1}|^p dx \right] \\ &\leq e_{16}(kp+1)^{\frac{p}{p-1}} \left[ \int_{\Omega} (u^+)^{(k+1)p} dx + 1 \right], \end{aligned}$$

equivalently

$$\|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}^p \leq (kp+1)^{\frac{1}{p-1}} (k+1)^p e_{17} \left[ \int_{\Omega} (u^+)^{(k+1)p} dx + 1 \right].$$

By Sobolev's embedding theorem a positive constant  $e_{18}$  exists such that

$$\|(u^+)^{k+1}\|_{L^{p^*}(\Omega)} \leq e_{18} \|(u^+)^{k+1}\|_{W^{1,p}(\Omega)},$$

where  $p^* = \frac{Np}{N-p}$  if  $1 < p < N$  and  $p^* = 2p$  if  $p = N$ . We get

$$\begin{aligned} &\|u^+\|_{L^{(k+1)p^*}(\Omega)} \\ &= \|(u^+)^{k+1}\|_{L^{p^*}(\Omega)}^{\frac{1}{k+1}} \\ &\leq e_{18}^{\frac{1}{k+1}} \|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}^{\frac{1}{k+1}} \\ &\leq e_{18}^{\frac{1}{k+1}} \left( (kp+1)^{\frac{1}{(p-1)p}} (k+1) \right)^{\frac{1}{k+1}} e_{17}^{\frac{1}{(k+1)p}} \left[ \int_{\Omega} (u^+)^{(k+1)p} dx + 1 \right]^{\frac{1}{(k+1)p}}. \end{aligned}$$

Since  $\left( (kp+1)^{\frac{1}{(p-1)p}} (k+1) \right)^{\frac{1}{k+1}} \geq 1$  and  $\lim_{k \rightarrow \infty} \left( (kp+1)^{\frac{1}{(p-1)p}} (k+1) \right)^{\frac{1}{k+1}} = 1$ ,

there exists a constant  $e_{19} > 1$  such that  $\left( (kp+1)^{\frac{1}{(p-1)p}} (k+1) \right)^{\frac{1}{k+1}} \leq e_{19}^{\frac{1}{\sqrt{k+1}}}$ . We obtain

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \leq e_{18}^{\frac{1}{k+1}} e_{19}^{\frac{1}{\sqrt{k+1}}} e_{17}^{\frac{1}{(k+1)p}} \left[ \int_{\Omega} (u^+)^{(k+1)p} dx + 1 \right]^{\frac{1}{(k+1)p}}. \quad (4.10)$$

Now, we will use the bootstrap arguments similarly as in the proof of [8, Lemma 3.2] starting with  $(k_1+1)p = p^*$  to get

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \leq c(k)$$

for any finite number  $k > 0$  which shows that  $u^+ \in L^r(\Omega)$  for any  $r \in (1, \infty)$ . To prove the uniform estimate with respect to  $k$  we argue as follows. If there is a sequence  $k_n \rightarrow \infty$  such that

$$\int_{\Omega} (u^+)^{(k_n+1)p} dx \leq 1,$$

we have immediately

$$\|u^+\|_{L^\infty(\Omega)} \leq 1,$$

(cf. the proof of [8, Lemma 3.2]). In the opposite case there exists  $k_0 > 0$  such that

$$\int_{\Omega} (u^+)^{(k+1)p} dx > 1$$

for any  $k \geq k_0$ . Then we conclude from (4.10)

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \leq e_{18}^{\frac{1}{k+1}} e_{19}^{\frac{1}{\sqrt{k+1}}} e_{20}^{\frac{1}{(k+1)p}} \|u^+\|_{L^{(k+1)p}}, \text{ for any } k \geq k_0, \quad (4.11)$$

where  $e_{20} = 2e_{17}$ . Choosing  $k := k_1$  such that  $(k_1 + 1)p = (k_0 + 1)p^*$  yields

$$\|u^+\|_{L^{(k_1+1)p^*}(\Omega)} \leq e_{18}^{\frac{1}{k_1+1}} e_{19}^{\frac{1}{\sqrt{k_1+1}}} e_{20}^{\frac{1}{(k_1+1)p}} \|u^+\|_{L^{(k_1+1)p}(\Omega)}.$$

Next, we can choose  $k_2$  in (4.11) such that  $(k_2 + 1)p = (k_1 + 1)p^*$  to get

$$\begin{aligned} \|u^+\|_{L^{(k_2+1)p^*}(\Omega)} &\leq e_{18}^{\frac{1}{k_2+1}} e_{19}^{\frac{1}{\sqrt{k_2+1}}} e_{20}^{\frac{1}{(k_2+1)p}} \|u^+\|_{L^{(k_2+1)p}(\Omega)} \\ &= e_{18}^{\frac{1}{k_2+1}} e_{19}^{\frac{1}{\sqrt{k_2+1}}} e_{20}^{\frac{1}{(k_2+1)p}} \|u^+\|_{L^{(k_1+1)p^*}(\Omega)}. \end{aligned}$$

By induction we obtain

$$\begin{aligned} \|u^+\|_{L^{(k_n+1)p^*}(\Omega)} &\leq e_{18}^{\frac{1}{k_n+1}} e_{19}^{\frac{1}{\sqrt{k_n+1}}} e_{20}^{\frac{1}{(k_n+1)p}} \|u^+\|_{L^{(k_n+1)p}(\Omega)} \\ &= e_{18}^{\frac{1}{k_n+1}} e_{19}^{\frac{1}{\sqrt{k_n+1}}} e_{20}^{\frac{1}{(k_n+1)p}} \|u^+\|_{L^{(k_{n-1}+1)p^*}(\Omega)}, \end{aligned}$$

where the sequence  $(k_n)$  is chosen such that  $(k_n + 1)p = (k_{n-1} + 1)p^*$  with  $k_0 > 0$ .

One easily verifies that  $k_n + 1 = \left(\frac{p^*}{p}\right)^n$ . Thus

$$\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} = e_{18}^{\sum_{i=1}^n \frac{1}{k_i+1}} e_{19}^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} e_{20}^{\sum_{i=1}^n \frac{1}{(k_i+1)p}} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)},$$

with  $r_n = (k_n + 1)p^* \rightarrow \infty$  as  $n \rightarrow \infty$ . Since  $\frac{1}{k_i+1} = \left(\frac{p}{p^*}\right)^i$  and  $\frac{p}{p^*} < 1$  there is a constant  $e_{21} > 0$  such that

$$\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \leq e_{21} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)} < \infty. \quad (4.12)$$

Let us assume that  $u^+ \notin L^\infty(\Omega)$ . Then there exist  $\eta > 0$  and a set  $A$  of positive measure in  $\Omega$  such that  $u^+(x) \geq e_{21} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta$  for  $x \in A$ . It follows that

$$\begin{aligned} \|u^+\|_{L^{(k_n+1)p^*}(\Omega)} &\geq \left( \int_A |u^+(x)|^{(k_n+1)p^*} \right)^{\frac{1}{(k_n+1)p^*}} \\ &\geq (e_{21} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta) |A|^{\frac{1}{(k_n+1)p^*}}. \end{aligned}$$

Passing to the limes inferior in the inequality above yields

$$\liminf_{n \rightarrow \infty} \|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \geq e_{21} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta,$$

which is a contradiction to (4.12) and hence,  $u^+ \in L^\infty(\Omega)$ . In a similar way one shows that  $u^- = \max\{-u, 0\} \in L^\infty(\Omega)$ . This proves  $u = u^+ - u^- \in L^\infty(\Omega)$ .  $\square$

**Remark 4.2.** Note that the finiteness of the integrals

$$\int_{\Omega} |\nabla(u^+)^{k+1}|^p dx, \quad \int_{\Omega} |(u^+)^{k+1}|^p dx$$

is shown in the end of the proof of Theorem 4.1 by a suitable choice of the parameter  $k$ . This is a typical proceeding in the use of the Moser iteration (see, e.g. [8]).

Let us now suppose an additional condition to the function  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  as follows.

(G4)  $g$  satisfies the condition

$$|g(x_1, s_1) - g(x_2, s_2)| \leq L \left[ |x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha \right],$$

for all pairs  $(x_1, s_1), (x_2, s_2)$  in  $\partial\Omega \times [-M_0, M_0]$ , where  $M_0$  is a positive constant and  $\alpha \in (0, 1]$ .

**Theorem 4.3.** Let the conditions (A1)–(A3), (F1)–(F3) and (G1)–(G4) be satisfied. Let  $u \in W^{1,p}(\Omega)$  be a solution of (1.1). Then  $u \in C^{1,\alpha}(\overline{\Omega})$ .

*Proof.* Theorem 4.1 implies  $u \in L^\infty(\Omega)$ . Moreover, we see at once that the assumptions (0.3a)–(0.3d) and (0.6) in [10] are satisfied which yields in view of [10, Theorem 2] the assertion.  $\square$

**Example 4.4.** Let  $A = -\Delta_p, 1 < p < \infty$ , be the negative  $p$ -Laplacian which is defined by

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad \text{where} \quad \nabla u = (\partial u / \partial x_1, \dots, \partial u / \partial x_N). \quad (4.13)$$

The coefficients  $a_i, i = 1, \dots, N$  are given by

$$a_i(x, s, \xi) = |\xi|^{p-2} \xi_i.$$

Thus, hypothesis (A1) is satisfied with  $k_0 = 0$  and  $c_0 = 1$ . Hypothesis (A2) is a consequence of the inequalities from the vector-valued function  $\xi \mapsto |\xi|^{p-2} \xi$  (see [2, Page 37]) and (A3) is satisfied with  $c_1 = 1$  and  $k_1 = 0$ . Our equation in (1.1) gets the form

$$\begin{aligned} -\Delta_p u &= f(x, u, \nabla u) && \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= g(x, u) && \text{on } \partial\Omega, \end{aligned} \quad (4.14)$$

where  $\frac{\partial u}{\partial \nu}$  means the outer normal derivative of  $u$  with respect to  $\partial\Omega$ . Theorem 4.1 and Theorem 4.3 ensure under the assumptions (F1)–(F3) and (G1)–(G4) that every solution  $u$  of (4.14) satisfies  $u \in L^\infty(\Omega)$  and  $u \in C^{1,\alpha}(\overline{\Omega})$ .

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