ON THE FUČIK SPECTRUM FOR THE p-LAPLACIAN WITH ROBIN BOUNDARY CONDITION

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Abstract. The aim of this paper is to study the Fučik spectrum of the p-Laplacian with Robin boundary condition given by

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned}$$

where $\beta \geq 0$. If $\beta = 0$, it reduces to the Fučik spectrum of the negative Neumann p-Laplacian. The existence of a first nontrivial curve $\mathcal{C}$ of this spectrum is shown and we prove some properties of this curve, e.g., $\mathcal{C}$ is Lipschitz continuous, decreasing and has a certain asymptotic behavior. A variational characterization of the second eigenvalue λ_2 of the Robin eigenvalue problem involving the p-Laplacian is also obtained.

1. Introduction

The Fučik spectrum of the negative p-Laplacian with a Robin boundary condition is defined as the set $\widehat{\Sigma}_p$ of $(a,b) \in \mathbb{R}^2$ such that

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

has a nontrivial solution. Here the domain $\Omega \subset \mathbb{R}^N$ is supposed to be bounded with a smooth boundary $\partial\Omega$. The notation $-\Delta_p u$ stands for the negative p-Laplacian of u, i.e., $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with $1 < p < +\infty$, while $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of u and β is a parameter belonging to $[0,+\infty)$. We also denote $u^{\pm} = \max\{\pm u, 0\}$. For $\beta = 0$, (1.1) becomes the Fučik spectrum of the negative Neumann p-Laplacian. Let us recall that $u \in W^{1,p}(\Omega)$ is a (weak) solution of (1.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \beta \int_{\partial\Omega} |u|^{p-2}uv \, d\sigma = \int_\Omega (a(u^+)^{p-1} - b(u^-)^{p-1})v \, dx, \tag{1.2}$$

for all $v \in W^{1,p}(\Omega)$. If $a = b = \lambda$, problem (1.1) reduces to

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

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which is known as the Robin eigenvalue problem for the p-Laplacian. As proved in [13], the first eigenvalue λ_1 of problem (1.3) is simple, isolated and can be characterized as follows

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma : \int_\Omega |u|^p dx = 1 \right\}.$$

It is also known that the eigenfunctions corresponding to λ_1 are of constant sign and belong to $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Throughout this paper, φ_1 denotes the eigenfunction of (1.3) associated to λ_1 which is normalized as $\|\varphi_1\|_{L^p(\Omega)} = 1$ and satisfies $\varphi_1 > 0$. Let us also recall that every eigenfunction of (1.3) corresponding to an eigenvalue $\lambda > \lambda_1$ must change sign.

We briefly describe the context of the Fučik spectrum related to problem (1.1). The Fučik spectrum was introduced by Fučik [11] in the case of the negative Laplacian in one dimension with periodic boundary conditions. He proved that this spectrum is composed of two families of curves emanating from the points (λ_k, λ_k) determined by the eigenvalues λ_k of the problem. Afterwards, many authors studied the Fučik spectrum Σ_2 for the negative Laplacian with Dirichlet boundary conditions (see [2, 4, 7, 14, 15, 18, 19, 24, 25] and the references therein). In this respect, we mention that Dancer [6] proved that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in Σ_2. De Figueiredo-Gossez [8] constructed a first nontrivial curve in Σ_2 through (λ_2, λ_2) and characterized it variationally. For $p \neq 2$ and in one dimension, Drábek [10] has shown that Σ_p has similar properties as in the linear case, i.e., $p = 2$. The Fučik spectrum Σ_p of the negative p-Laplacian with homogeneous Dirichlet boundary conditions in the general case $1 < p < +\infty$ and $N \geq 1$, that is

$$(a,b) \in \Sigma_p : \qquad \begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has been studied by Cuesta-de Figueiredo-Gossez [5], where the authors proved the existence of a first nontrivial curve through (λ_2, λ_2) and that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in Σ_p. For other results on Σ_p we refer to [20, 21, 22, 23].

The Fučik spectrum Θ_p of the negative p-Laplacian with homogeneous Neumann boundary condition, that is

$$(a,b) \in \Theta_p : \qquad \begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

was investigated in [1, 3, 17]. It is worth emphasizing that Arias-Campos-Gossez [3] pointed out an important difference between the cases $p \leq N$ and $p > N$ regarding the asymptotic properties of the first nontrivial curve in Θ_p. Note that the Fučik spectrum Θ_p is incorporated in problem (1.1) by taking $\beta = 0$. Finally, we mention the work of Martínez and Rossi [16] who considered the Fučik spectrum $\widetilde{\Sigma}_p$ associated to Steklov boundary condition, which is introduced by

$$(a,b) \in \widetilde{\Sigma}_p : \qquad \begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{on } \partial\Omega. \end{aligned}$$

As in the previous situations, they constructed a first nontrivial curve in $\widetilde{\Sigma}_p$ through (λ_2, λ_2), where λ_2 denotes the second eigenvalue of the Steklov eigenvalue problem, and studied its asymptotic behavior.

The aim of this paper is the study of the Fučik spectrum $\widehat{\Sigma}_p$ given in (1.1) for the negative p-Laplacian with Robin boundary condition. We are going to prove the existence of a first nontrivial curve $\mathcal{C}$ of this spectrum and show that it shares the same properties as in the cases of the other problems discussed above: Lipschitz continuity, strictly decreasing monotonicity and asymptotic behavior. It is a significant fact that the presence of the parameter β in problem (1.1) does not alter these basic properties. The main idea in studying the asymptotic behavior of the curve $\mathcal{C}$ is the use of a suitable equivalent norm related to β. A relevant consequence of the construction of the first nontrivial curve $\mathcal{C}$ in $\widehat{\Sigma}_p$ is the following variational characterization of the second eigenvalue λ_2 of (1.3):

$$\lambda_2 = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \left[\int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma\right], \tag{1.4}$$

where

$$\Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1,\ \gamma(1) = \varphi_1\},$$

with

$$S = \left\{u \in W^{1,p}(\Omega) : \int_\Omega |u|^p dx = 1\right\}. \tag{1.5}$$

The results presented in this paper complete the picture of the Fučik spectrum involving the p-Laplacian by adding in the case of Robin condition the information previously known for Dirichlet problem (see [5]), Steklov problem (see [16]), and homogeneous Neumann problem (see [3]). Actually, as already specified, the results given here for the Fučik spectrum (1.1) of the negative p-Laplacian with Robin boundary condition extend the ones known for the Fučik spectrum Θ_p under Neumann boundary condition by simply making $\beta = 0$.

Our approach is variational relying on the functional associated to problem (1.1), which is expressed on $W^{1,p}(\Omega)$ by

$$J(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma - \int_\Omega (a(u^+)^p + b(u^-)^p) dx.$$

It is clear that $J \in C^1(W^{1,p}(\Omega), \mathbb{R})$ and the critical points of J coincide with the weak solutions of problem (1.1). In comparison with the corresponding functionals related to the Fučik spectrum for the Dirichlet and Steklov problems, the functional J exhibits an essential difference because its expression does not contain the norm of the space $W^{1,p}(\Omega)$, and it is also different from the functional used to treat the Neumann problem because it contains the additional boundary term involving β. However, in our proofs various ideas and techniques are worked out on the pattern of [3], [5], [16].

The rest of the paper is organized as follows. Section 2 is devoted to the determination of elements of $\widehat{\Sigma}_p$ by means of critical points of a suitable functional. Section 3 sets forth the construction of the first nontrivial curve $\mathcal{C}$ in $\widehat{\Sigma}_p$ and the variational characterization of the second eigenvalue λ_2 for (1.3). Section 4 presents the basic properties of $\mathcal{C}$.

2. The spectrum $\widehat{\Sigma}_p$ through critical points

The aim of this section is to determine elements of the Fučik spectrum $\widehat{\Sigma}_p$ defined in problem (1.1). They are found by critical points of a functional that is constructed by means of the Robin problem (1.1). To this end we follow certain ideas in [5] and [16] developed for problems with Dirichlet and Steklov boundary conditions.

For a fixed $s \in \mathbb{R}$, $s \geq 0$, and corresponding to $\beta \geq 0$ given in problem (1.1), we introduce the functional $J_s : W^{1,p}(\Omega) \to \mathbb{R}$ by

$$J_s(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma - s \int_\Omega (u^+)^p dx,$$

thus $J_s \in C^1(W^{1,p}(\Omega), \mathbb{R})$. The set S introduced in (1.5) is a smooth submanifold of $W^{1,p}(\Omega)$, and thus $\widetilde{J}_s = J_s|_S$ is a C^1 function in the sense of manifolds. We note that $u \in S$ is a critical point of $\widetilde{J}_s$ (in the sense of manifolds) if and only if there exists $t \in \mathbb{R}$ such that

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \beta \int_{\partial\Omega} |u|^{p-2} uv \, d\sigma - s \int_\Omega (u^+)^{p-1} v \, dx \\ &= t \int_\Omega |u|^{p-2} uv \, dx, \quad \forall v \in W^{1,p}(\Omega).\end{aligned} \tag{2.1}$$

Now we describe the relationship between the critical points of $\widetilde{J}_s$ and the spectrum $\widehat{\Sigma}_p$.

Lemma 2.1. *Given a number s, one has that $(s + J_s(u), J_s(u)) \in \mathbb{R}^2$ belongs to the spectrum $\widehat{\Sigma}_p$ if and only if there exists a critical point $u \in S$ of $\widetilde{J}_s$ such that $t = J_s(u)$.*

Proof. The definition in (1.2) for the weak solution shows that $(t+s, t) \in \widehat{\Sigma}_p$ if and only if there is $u \in S$ that solves the Robin problem

$$\begin{aligned} -\Delta_p u &= (t+s)(u^+)^{p-1} - t(u^-)^{p-1} && \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= -\beta |u|^{p-2} u && \text{on } \partial\Omega, \end{aligned}$$

which means exactly (2.1). Inserting $v = u$ in (2.1) yields $t = J_s(u)$, as required. □

Lemma 2.1 enables us to find points in $\widehat{\Sigma}_p$ through the critical points of $\widetilde{J}_s$. In order to implement this, first we look for minimizers of $\widetilde{J}_s$.

Proposition 2.2. *There hold:*
- (i) *the first eigenfunction φ_1 is a global minimizer of $\widetilde{J}_s$;*
- (ii) *the point $(\lambda_1, \lambda_1 - s) \in \mathbb{R}^2$ belongs to $\widehat{\Sigma}_p$.*

Proof. (i) Since $\beta, s \geq 0$, using the characterization of λ_1 we have

$$\begin{aligned}\widetilde{J}_s(u) &= \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma - s \int_\Omega (u^+)^p dx \\ &\geq \lambda_1 \int_\Omega |u|^p dx - s \int_\Omega (u^+)^p dx \geq \lambda_1 - s = J_s(\varphi_1), \quad \forall u \in S.\end{aligned}$$

(ii) On the basis of (i), we can apply Lemma 2.1. □

Next we produce a second critical point of $\widetilde{J}_s$ as a local minimizer.

Proposition 2.3. *There hold:*
- (i) *the negative eigenfunction $-\varphi_1$ is a strict local minimizer of $\widetilde{J}_s$;*
- (ii) *the point $(\lambda_1 + s, \lambda_1) \in \mathbb{R}^2$ belongs to $\widehat{\Sigma}_p$.*

Proof. (i) Arguing indirectly, let us suppose that there exists a sequence $(u_n) \subset S$ with $u_n \neq -\varphi_1, u_n \to -\varphi_1$ in $W^{1,p}(\Omega)$ and $\widetilde{J}_s(u_n) \leq \lambda_1 = \widetilde{J}_s(-\varphi_1)$. If $u_n \leq 0$ for a.a. $x \in \Omega$, we obtain

$$\widetilde{J}_s(u_n) = \int_\Omega |\nabla u_n|^p dx + \beta \int_{\partial\Omega} |u_n|^p d\sigma > \lambda_1,$$

because $u_n \neq -\varphi_1$ and $u_n \neq \varphi_1$, which contradicts the assumption $\widetilde{J}_s(u_n) \leq \lambda_1$. Consider now the complementary situation. Hence u_n changes sign whenever n is sufficiently large, thereby we can set

$$w_n = \frac{u_n^+}{\|u_n^+\|_{L^p(\Omega)}} \quad \text{and} \quad r_n = \|\nabla w_n\|^p_{L^p(\Omega)} + \beta \|w_n\|^p_{L^p(\partial\Omega)}. \tag{2.2}$$

We claim that, along a relabeled subsequence, $r_n \to +\infty$ as $n \to \infty$. Suppose by contradiction that (r_n) is bounded. This implies through (2.2) that (w_n) is bounded in $W^{1,p}(\Omega)$, so there exists a subsequence denoted again by (w_n) such that $w_n \to w$ in $L^p(\Omega)$, for some $w \in W^{1,p}(\Omega)$. Since $\|w_n\|_{L^p(\Omega)} = 1$ and $w_n \geq 0$ a.e., we get $\|w\|_{L^p(\Omega)} = 1$ and $w \geq 0$. This contradicts the assumption that $u_n \to -\varphi_1$ in $L^p(\Omega)$, thus proving the claim.
On the other hand, from (2.2) and by using the variational characterization of λ_1, we infer that

$$\begin{aligned}\widetilde{J}_s(u_n) &= (r_n - s) \int_\Omega |u_n^+|^p dx + \int_\Omega |\nabla u_n^-|^p dx + \beta \int_{\partial\Omega} |u_n^-|^p d\sigma \\ &\geq (r_n - s) \int_\Omega |u_n^+|^p dx + \lambda_1 \int_\Omega (u_n^-)^p dx,\end{aligned}$$

whereas the choice of (u_n) gives

$$\widetilde{J}_s(u_n) \leq \lambda_1 = \lambda_1 \int_\Omega (u_n^+)^p dx + \lambda_1 \int_\Omega (u_n^-)^p dx.$$

Combining the inequalities above results in

$$(\lambda_1 - r_n + s) \int_\Omega (u_n^+)^p dx \geq 0,$$

therefore $\lambda_1 \geq r_n - s$. This is against the unboundedness of (r_n), which completes the proof of (i). Part (ii) follows from Lemma 2.1 because $J_s(-\varphi_1) = \lambda_1$. □

Using the two local minima obtained in Propositions 2.2 and 2.3, we seek for a third critical point of $\widetilde{J}_s$ via a version of the Mountain-Pass Theorem on C^1-manifolds (see, e.g., [12, Theorem 3.2]). First, we check the Palais-Smale condition for $\widetilde{J}_s$ on the manifold S.

Lemma 2.4. *The functional $\widetilde{J}_s : S \to \mathbb{R}$ satisfies the Palais-Smale condition on S in the sense of manifolds.*

Proof. Let $(u_n) \subset S$ be a sequence provided $(J_s(u_n))$ is bounded and $\|\widetilde{J}_s'(u_n)\|_* \to 0$ as $n \to \infty$, which means that there exists a sequence $(t_n) \subset \mathbb{R}$ such that

$$\begin{aligned}\Bigg| &\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v \, dx + \beta \int_{\partial\Omega} |u_n|^{p-2} u_n v \, d\sigma \\ &- s \int_\Omega (u_n^+)^{p-1} v \, dx - t_n \int_\Omega |u_n|^{p-2} u_n v \, dx \Bigg| \leq \varepsilon_n \|v\|_{W^{1,p}(\Omega)},\end{aligned} \tag{2.3}$$

for all $v \in W^{1,p}(\Omega)$ and with $\varepsilon_n \to 0^+$. Note that $J_s(u_n) \geq \|\nabla u_n\|^p_{L^p(\Omega)} - s$. Since $(u_n) \in S$ and $(J_s(u_n))$ is bounded, we derive that (u_n) is bounded in $W^{1,p}(\Omega)$. Thus, along a relabeled subsequence we may suppose that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, $u_n \to u$ in $L^p(\Omega)$ and $u_n \to u$ in $L^p(\partial\Omega)$. Taking $v = u_n$ in (2.3) and using again $(u_n) \subset S$ shows that the sequence (t_n) is bounded. Then, if we choose $v = u_n - u$, it follows that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u)dx \to 0 \quad \text{as } n \to \infty.$$

At this point, the $(S)_+$-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ enables us to conclude that $u_n \to u$ in $W^{1,p}(\Omega)$. □

Now we obtain, in addition to φ_1 and $-\varphi_1$, a third critical point of $\widetilde{J}_s$ on S.

Proposition 2.5. *There hold:*

(i)

$$c(s) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} J_s(u), \tag{2.4}$$

where

$$\Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\},$$

is a critical value of $\widetilde{J}_s$ satisfying $c(s) > \max\{\widetilde{J}_s(-\varphi_1), \widetilde{J}_s(\varphi_1)\} = \lambda_1$. In particular, there exists a critical point of $\widetilde{J}_s$ that is different from $-\varphi_1$ and φ_1.

(ii) *The point $(s + c(s), c(s))$ belongs to $\widehat{\Sigma}_p$.*

Proof. (i) By Proposition 2.3 we know that $-\varphi_1$ is a strict local minimizer of $\widetilde{J}_s$ with $\widetilde{J}_s(-\varphi_1) = \lambda_1$, while Proposition 2.2 ensures that φ_1 is a global minimizer of $\widetilde{J}_s$ with $\widetilde{J}_s(\varphi_1) = \lambda_1 - s$. Then we can show that

$$\inf\{\widetilde{J}_s(u) : u \in S \text{ and } \|u - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \varepsilon\} > \max\{\widetilde{J}_s(-\varphi_1), \widetilde{J}_s(\varphi_1)\} = \lambda_1,$$

whenever $\varepsilon > 0$ is sufficiently small. The proof that the inequality above is strict can be done as in [5, Lemma 2.9] on the basis of Ekeland's variational principle. In order to fulfill the mountain-pass geometry we choose $\varepsilon > 0$ even smaller if necessary to have $2\|\varphi_1\|_{W^{1,p}(\Omega)} = \|\varphi_1 - (-\varphi_1)\|_{W^{1,p}(\Omega)} > \varepsilon$. Since $\widetilde{J}_s : S \to \mathbb{R}$ satisfies the Palais-Smale condition on the manifold S as shown in Lemma 2.4, we may invoke the version of Mountain-Pass Theorem on manifolds (see, e.g., [12, Theorem 3.2]). This guarantees that $c(s)$ introduced in (2.4) is a critical value of $\widetilde{J}_s$ with $c(s) > \lambda_1$, providing a critical point different from $-\varphi_1$ and φ_1.

(ii) Thanks to Lemma 2.1 and part (i), we infer that $(s + c(s), c(s)) \in \widehat{\Sigma}_p$. □

3. The first nontrivial curve

The results in Section 2 permit to determine the beginning of the spectrum $\widehat{\Sigma}_p$. We start by establishing that the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are isolated in $\widehat{\Sigma}_p$. This is known from [5, Proposition 3.4] for Dirichlet problems and from [16, Proposition 3.1] for Steklov problems.

Proposition 3.1. *There exists no sequence $(a_n, b_n) \in \widehat{\Sigma}_p$ with $a_n > \lambda_1$ and $b_n > \lambda_1$ such that $(a_n, b_n) \to (a, b)$ with $a = \lambda_1$ or $b = \lambda_1$.*

Proof. Proceeding indirectly, assume there exist sequences $(a_n, b_n) \in \widehat{\Sigma}_p$ and $(u_n) \subset W^{1,p}(\Omega)$ with the properties: $a_n \to \lambda_1$, $b_n \to b$, $a_n > \lambda_1$, $b_n > \lambda_1$, $\|u_n\|_{L^p(\Omega)} = 1$ and

$$\begin{aligned} -\Delta_p u_n &= a_n (u_n^+)^{p-1} - b_n (u_n^-)^{p-1} && \text{in } \Omega, \\ |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} &= -\beta |u_n|^{p-2} u_n && \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

If we test (3.1) with $v = u_n$ (see (1.2)), we get

$$\|\nabla u_n\|^p_{L^p(\Omega)} = a_n \int_\Omega (u_n^+)^p \, dx + b_n \int_\Omega (u_n^-)^p \, dx - \beta \int_{\partial\Omega} |u_n|^p \, d\sigma \le a_n + b_n,$$

which proves the boundedness of (u_n) in $W^{1,p}(\Omega)$. Hence, along a subsequence, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$. Now, testing (3.1) with $\varphi = u_n - u$, we infer that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla(u_n - u) \, dx = 0.$$

The $(S)_+$-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ yields that $u_n \to u$ in $W^{1,p}(\Omega)$. Thus, u is a solution of the equation

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \\ &= \lambda_1 \int_\Omega (u^+)^{p-1} v \, dx - b \int_\Omega (u^-)^{p-1} v \, dx - \beta \int_{\partial\Omega} |u|^{p-2} uv \, d\sigma, \end{aligned} \tag{3.2}$$

for all $v \in W^{1,p}(\Omega)$. Inserting $v = u^+$ in (3.2) leads to

$$\int_\Omega |\nabla u^+|^p \, dx = \lambda_1 \int_\Omega (u^+)^p \, dx - \beta \int_{\partial\Omega} (u^+)^p d\sigma.$$

This, in conjunction with the characterization of λ_1 in Section 1 and since $\|u\|_{L^p(\Omega)} = 1$, ensures that either $u^+ = 0$ or $u^+ = \varphi_1$. If $u^+ = 0$, then $u \le 0$ and (3.2) implies that u is an eigenfunction. Recalling that λ_1 is the only eigenfunction that does not change sign, we deduce that $u = -\varphi_1$ (see [13] and also Proposition 4.1). This renders that (u_n) converges either to φ_1 or to $-\varphi_1$ in $L^p(\Omega)$, which forces to have

$$\text{either} \quad |\{x \in \Omega : u_n(x) < 0\}| \to 0 \quad \text{or} \quad |\{x \in \Omega : u_n > 0\}| \to 0, \tag{3.3}$$

respectively, where $|\cdot|$ denotes the Lebesgue measure. Indeed, assuming for instance $u_n \to \varphi_1$ in $L^p(\Omega)$, since for any compact subset $K \subset \Omega$ there holds

$$\int_{\{u_n<0\}\cap K} |u_n - \varphi_1|^p \, dx \ge \int_{\{u_n<0\}\cap K} \varphi_1^p \, dx \ge C|\{u_n < 0\} \cap K|,$$

with a constant $C > 0$, it is seen that the first assertion in (3.3) is fulfilled.

On the other hand, using $v = u_n^+$ as test function for (3.1) in conjunction with the Hölder inequality and the continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, with $p < q \le p^*$, we obtain the estimate

$$\begin{aligned} \int_\Omega |\nabla u_n^+|^p dx + \int_\Omega (u_n^+)^p dx &= a_n \int_\Omega (u_n^+)^p dx - \beta \int_{\partial\Omega} (u_n^+)^p d\sigma + \int_\Omega (u_n^+)^p dx \\ &\le (a_n + 1) \int_\Omega (u_n^+)^p dx \\ &\le (a_n + 1) C |\{x \in \Omega : u_n(x) > 0\}|^{1-\frac{p}{q}} \|u_n^+\|^p_{W^{1,p}(\Omega)}, \end{aligned}$$

with a constant $C > 0$. We infer that

$$|\{x \in \Omega : u_n(x) > 0\}|^{1-\frac{p}{q}} \geq (a_n + 1)^{-1} C^{-1}.$$

and in the same way,

$$|\{x \in \Omega : u_n(x) < 0\}|^{1-\frac{p}{q}} \geq (b_n + 1)^{-1} C^{-1}.$$

Since (a_n, b_n) does not belong to the trivial lines of $\widehat{\Sigma}_p$, we have that u_n changes sign. Hence we reach a contradiction with (3.3), which completes the proof. □

The following auxiliary fact is helpful to link with the results established in Section 2.

Lemma 3.2. *For every $r > \inf_S J_s = \lambda_1 - s$, each connected component of $\{u \in S : J_s(u) < r\}$ contains a critical point, in fact a local minimizer of $\widetilde{J}_s$.*

Proof. Let C be a connected component of $\{u \in S : J_s(u) < r\}$ and denote $d = \inf\{J_s(u) : u \in \overline{C}\}$. We claim that there exists $u_0 \in \overline{C}$ such that $\widetilde{J}_s(u_0) = d$. To this end, let $(u_n) \subset \overline{C}$ be a sequence such that $\widetilde{J}_s(u_n) \leq d + \frac{1}{n^2}$. Applying Ekeland's variational principle to $\widetilde{J}_s$ on $\overline{C}$ provides a sequence $(v_n) \subset \overline{C}$ such that

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u_n), \tag{3.4}$$

$$\|u_n - v_n\|_{W^{1,p}(\Omega)} \leq \frac{1}{n}, \tag{3.5}$$

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(v) + \frac{1}{n}\|v - v_n\|_{W^{1,p}(\Omega)}, \quad \forall v \in \overline{C}. \tag{3.6}$$

If n is sufficiently large, by (3.4) we obtain

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u_n) \leq d + \frac{1}{n^2} < r.$$

Moreover, owing to (3.6), it can be shown that (v_n) is a Palais-Smale sequence for $\widetilde{J}_s$. Then Lemma 2.4 and (3.5) ensure that, up to a relabeled subsequence, $u_n \to u_0$ in $W^{1,p}(\Omega)$ with $u_0 \in \overline{C}$ and $\widetilde{J}_s(v) = d$.

We note that $u_0 \notin \partial C$ because otherwise the maximality of C as a connected component would be contradicted, so u_0 is a local minimizer of $\widetilde{J}_s$ and we are done. □

Recall from Proposition 2.5 that it was constructed a curve $(s + c(s), c(s)) \in \widehat{\Sigma}_p$ for $s \geq 0$. As $\widehat{\Sigma}_p$ is symmetric with respect to the diagonal, we can complete it with its symmetric part obtaining the following curve in $\widehat{\Sigma}_p$:

$$\mathcal{C} := \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\}. \tag{3.7}$$

The next result points out that $\mathcal{C}$ is the first nontrivial curve in $\widehat{\Sigma}_p$.

Theorem 3.3. *Let $s \geq 0$. Then $(s + c(s), c(s)) \in \mathcal{C}$ is the first point in the intersection between $\widehat{\Sigma}_p$ and the ray $(s, 0) + t(1, 1)$, $t > \lambda_1$.*

Proof. Assume, by contradiction, the existence of a point $(s + \mu, \mu) \in \widehat{\Sigma}_p$ with $\lambda_1 < \mu < c(s)$. Proposition 3.1 and the fact that $\widehat{\Sigma}_p$ is closed enable us to suppose that μ is the minimum number with the required property. By virtue of Lemma 2.1, μ is a critical value of the functional J_s and there is no critical value of $\widetilde{J}_s$ in the interval (λ_1, μ). We complete the proof by reaching a contradiction to the

definition of $c(s)$ in (2.4). To this end, it suffices to construct a path in Γ along which there holds $\widetilde{J}_s \leq \mu$.
Let $u \in S$ be a critical point of $\widetilde{J}_s$ with $\widetilde{J}_s(u) = \mu$. Then u fulfills

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx =&(s+\mu)\int_\Omega (u^+)^{p-1} v\, dx - \mu \int_\Omega (u^-)^{p-1} v\, dx\\ &- \beta \int_{\partial\Omega} |u|^{p-2} uv\, d\sigma, \quad \forall v \in W^{1,p}(\Omega).\end{aligned}$$

Setting $v = u^+$ and $v = -u^-$ yields

$$\int_\Omega |\nabla u^+|^p\, dx = (s+\mu)\int_\Omega (u^+)^p\, dx - \beta \int_{\partial\Omega} (u^+)^p d\,\sigma \tag{3.8}$$

and

$$\int_\Omega |\nabla u^-|^p\, dx = \mu \int_\Omega (u^-)^p\, dx - \beta \int_{\partial\Omega} (u^-)^p\, d\sigma, \tag{3.9}$$

respectively. Since u changes sign (see Proposition 4.1), the following paths are well defined on S:

$$\begin{aligned} u_1(t) &= \frac{(1-t)u + tu^+}{\|(1-t)u + tu^+\|_{L^p(\Omega)}},\\ u_2(t) &= \frac{(1-t)u^+ + tu^-}{\|(1-t)u^+ + tu^-\|_{L^p(\Omega)}},\\ u_3(t) &= \frac{-tu^- + (1-t)u}{\|-tu^- + (1-t)u\|_{L^p(\Omega)}}, \end{aligned}$$

for all $t \in [0,1]$. By means of direct calculations based on (3.8) and (3.9) we infer that

$$\widetilde{J}_s(u_1(t)) = \widetilde{J}_s(u_3(t)) = \mu, \quad \text{for all } t \in [0,1]$$

and

$$\widetilde{J}_s(u_2(t)) = \mu - \frac{st^p \|u^-\|_{L^p(\Omega)}}{\|(1-t)u^+ + tu^-\|_{L^p(\Omega)}} \leq \mu, \quad \text{for all } t \in [0,1].$$

Due to the minimality property of μ, the only critical points of $\widetilde{J}_s$ in the set $\{w \in S : \widetilde{J}_s(w) < \mu - s\}$ are φ_1 and possibly $-\varphi_1$ provided $\mu - s > \lambda_1$. We note that, because $u^-/\|u^-\|_{L^p(\Omega)}$ does not change sign, it is not a critical point of $\widetilde{J}_s$. Therefore, there exists a C^1 path $\alpha : [-\varepsilon, \varepsilon] \to S$ with $\alpha(0) = u^-/\|u^-\|_{L^p(\Omega)}$ and $d/dt \widetilde{J}_s(\alpha(t))|_{t=0} \neq 0$. Using this path and observing from (3.9) that $\widetilde{J}_s(u^-/\|u^-\|_{L^p(\Omega)}) = \mu - s$, we can move from $u^-/\|u^-\|_{L^p(\Omega)}$ to a point v with $\widetilde{J}_s(v) < \mu - s$. Applying Lemma 3.2, we find that the connected component of $\{w \in S : \widetilde{J}_s(w) < \mu - s\}$ containing v crosses $\{\varphi_1, -\varphi_1\}$. Let us say it passes through φ_1, otherwise the reasoning is the same employing $-\varphi_1$. Consequently, there is a path $u_4(t)$ from $u^-/\|u^-\|_{L^p(\Omega)}$ to φ_1 within the set $\{w \in S : \widetilde{J}_s(w) < \mu - s\}$. Then the path $-u_4(t)$ joins $-u^-/\|u^-\|_{L^p(\Omega)}$ and $-\varphi_1$ and, since $u_4(t) \in S$, we have

$$\widetilde{J}_s(-u_4(t)) \leq \widetilde{J}_s(u_4(t)) + s < \mu - s + s = \mu \ \text{ for all } t.$$

Connecting $u_1(t)$, $u_2(t)$ and $u_4(t)$, we construct a path joining u and φ_1, and joining $u_3(t)$ and $-u_4(t)$ we get a path which connects u and $-\varphi_1$. These yield a path $\gamma(t)$

on S joining φ_1 and $-\varphi_1$. Furthermore, in view of the discussion above, it turns out $\widetilde{J}_s(\gamma(t)) \leq \mu$ for all t. This proves the theorem. □

Corollary 3.4. *The second eigenvalue λ_2 of (1.3) has the variational characterization given in (1.4).*

Proof. Theorem 3.3 for $s = 0$ ensures that $c(0) = \lambda_2$. The conclusion now follows by applying Proposition 2.5 (i) with $s = 0$. □

4. Properties of the first curve

The following proposition establishes an important sign property related to the curve $\mathcal{C}$ in (3.7).

Proposition 4.1. *Let $(a_0, b_0) \in \mathcal{C}$ and $a, b \in L^\infty(\Omega)$ satisfy $\lambda_1 \leq a(x) \leq a_0$, $\lambda_1 \leq b(x) \leq b_0$ for a.a. $x \in \Omega$ such that $\lambda_1 < a(x)$ and $\lambda_1 < b(x)$ on subsets of positive measure. Then any nontrivial solution u of*

$$\begin{aligned} -\Delta_p u &= a(x)(u^+)^{p-1} - b(x)(u^-)^{p-1} && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

changes sign in Ω.

Proof. Let u be a nontrivial solution of equation (4.1). Then, $-u$ is a nontrivial solution of

$$\begin{aligned} -\Delta_p z &= b(x)(z^+)^{p-1} - a(x)(z^-)^{p-1} && \text{in } \Omega,\\ |\nabla z|^{p-2}\frac{\partial z}{\partial \nu} &= -\beta |z|^{p-2}z && \text{on } \partial\Omega, \end{aligned}$$

hence, we can suppose that the point $(a_0, b_0) \in \mathcal{C}$ is such that $a_0 \geq b_0$.
We argue by contradiction and assume that u does not change sign in Ω. Without loss of generality, we may admit that $u \geq 0$ a.e. in Ω, so u is a solution of the Robin weighted eigenvalue problem with weight $a(x)$:

$$\begin{aligned} -\Delta_p u &= a(x)u^{p-1} && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta u^{p-1} && \text{on } \partial\Omega. \end{aligned}$$

It means that u is an eigenfunction corresponding to the eigenvalue 1 for this problem. Recall that the first eigenvalue $\lambda_1(a)$ of the above weighted problem is expressed as

$$\lambda_1(a) = \inf_{\substack{v \in W^{1,p}(\Omega)\\ v \not\equiv 0}} \frac{\int_\Omega |\nabla v|^p dx + \beta \int_{\partial\Omega} |v|^p d\sigma}{\int_\Omega a(x)|v|^p dx}.$$

The fact that $u \geq 0$ entails $\lambda_1(a) = 1$ because the only eigenfunction whose eigenfunctions do not change sign is $\lambda_1(a)$ (see [13]). Then the hypothesis that $\lambda_1 < a(x)$ on a set of positive measure leads to the contradiction

$$1 = \frac{\int_\Omega |\nabla \varphi_1|^p dx + \beta \int_{\partial\Omega} |\varphi_1|^p d\sigma}{\lambda_1} > \frac{\int_\Omega |\nabla \varphi_1|^p dx + \beta \int_{\partial\Omega} |\varphi_1|^p d\sigma}{\int_\Omega a(x)\varphi_1^p dx} \geq \lambda_1(a) = 1,$$

which completes the proof. □

Proposition 4.2. *The curve $s \mapsto (s + c(s), c(s))$ is Lipschitz continuous and decreasing.*

Proof. If $s_1 < s_2$, then it follows $\widetilde{J}_{s_1}(u) \geq \widetilde{J}_{s_2}(u)$ for all $u \in S$, which ensures that $c(s_1) \geq c(s_2)$. For every $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$\max_{u\in\gamma[-1,1]} \widetilde{J}_{s_2}(u) \leq c(s_2) + \varepsilon,$$

hence

$$0 \leq c(s_1) - c(s_2) \leq \max_{u\in\gamma[-1,1]} \widetilde{J}_{s_1}(u) - \max_{u\in\gamma[-1,1]} \widetilde{J}_{s_2}(u) + \varepsilon.$$

Taking $u_0 \in \gamma[-1, 1]$ such that

$$\max_{u\in\gamma[-1,1]} \widetilde{J}_{s_1}(u) = \widetilde{J}_{s_1}(u_0)$$

yields

$$0 \leq c(s_1) - c(s_2) \leq \widetilde{J}_{s_1}(u_0) - \widetilde{J}_{s_2}(u_0) + \varepsilon = s_1 - s_2 + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this ensures that $s \mapsto (s+c(s), c(s))$ is Lipschitz continuous. In order to prove that the curve is decreasing, it suffices to argue for $s > 0$. Let $0 < s_1 < s_2$. Then, since $(s_1 + c(s_1), c(s_1)), (s_2 + c(s_2), c(s_2)) \in \widehat{\Sigma}_p$, Theorem 3.3 implies that $s_1 + c(s_1) < s_2 + c(s_2)$. On the other hand, as already remarked, there holds $c(s_1) \geq c(s_2)$, which completes the proof. □

Next we investigate the asymptotic behavior of the curve $\mathcal{C}$.

Theorem 4.3. *Let $p \leq N$. Then the limit of $c(s)$ as $s \to +\infty$ is λ_1.*

Proof. Let us proceed by contradiction and suppose that $c(s)$ does not converge to λ_1 as $s \to +\infty$. Then there exists $\delta > 0$ such that

$$\max_{u\in\gamma[-1,1]} \widetilde{J}_s(u) \geq \lambda_1 + \delta \quad \text{for all } \gamma \in \Gamma \text{ and all } s \geq 0.$$

Since $p \leq N$, we can choose a function $\psi \in W^{1,p}(\Omega)$ which is unbounded from above. Then we define $\gamma \in \Gamma$ by

$$\gamma(t) = \frac{t\varphi_1 + (1 - |t|)\psi}{\|t\varphi_1 + (1 - |t|)\psi\|_{L^p(\Omega)}}, \quad t \in [-1, 1].$$

For every $s > 0$, let $t_s \in [-1, 1]$ satisfy

$$\max_{t\in[-1,1]} \widetilde{J}_s(\gamma(t)) = \widetilde{J}_s(\gamma(t_s)).$$

Denoting $v_s = t_s\varphi_1 + (1 - |t_s|)\psi$, we infer that

$$\int_\Omega |\nabla v_s|^p dx + \beta \int_{\partial\Omega} |v_s|^p d\sigma - s \int_\Omega (v_s^+)^p dx \geq (\lambda_1 + \delta) \int_\Omega |v_s|^p dx. \tag{4.2}$$

Letting $s \to +\infty$, we can assume along a subsequence that $t_s \to \widetilde{t} \in [-1, 1]$. The family v_s being bounded in $W^{1,p}(\Omega)$, from (4.2) one sees that

$$\int_\Omega (v_s^+)^p dx \to 0 \quad \text{as } s \to +\infty,$$

which forces

$$\widetilde{t}\varphi_1 + (1 - |\widetilde{t}|)\psi \leq 0.$$

Due to the choice of ψ, this is impossible unless $\widetilde{t} = -1$. Passing to the limit in (4.2) as $s \to +\infty$ and using $\widetilde{t} = -1$, we arrive at the contradiction $\delta \leq 0$, so the proof is complete.

□

It remains to study the asymptotic properties of the curve $\mathcal{C}$ when $p > N$. For $\beta = 0$, problem (1.1) becomes a Neumann problem with homogeneous boundary condition that was studied in [3]. Therein, it is shown that

$$\lim_{s\to+\infty} c(s) = \begin{cases} \lambda_1 = 0 & \text{if } p \le N \\ \widetilde{\lambda} & \text{if } p > N, \end{cases}$$

where

$$\widetilde{\lambda} = \inf\left\{ \int_\Omega |\nabla u|^p dx : u \in W^{1,p}(\Omega), \|u\|_{L^p(\Omega)} = 1 \text{ and } u \text{ vanishes somewhere in } \overline{\Omega}\right\}.$$

Therefore, we only have to treat the case $\beta > 0$. In this respect, the key idea is to work with an adequate equivalent norm on the space $W^{1,p}(\Omega)$. So, for $\beta > 0$ we introduce the norm

$$\|u\|_\beta = \|\nabla u\|_{L^p(\Omega)} + \beta\|u\|_{L^p(\partial\Omega)}, \tag{4.3}$$

which is an equivalent norm on $W^{1,p}(\Omega)$ (see also Deng [9, Theorem 2.1]). Then we have the following.

Theorem 4.4. *Let $\beta > 0$ and $p > N$. Then the limit of $c(s)$ as $s \to +\infty$ is*

$$\overline{\lambda} = \inf_{u\in L} \max_{r\in\mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u)|^p dx + \beta\int_{\partial\Omega} |r\varphi_1 + u|^p d\sigma}{\int_\Omega |r\varphi_1 + u|^p dx},$$

where

$$L = \{u \in W^{1,p}(\Omega) : u \text{ vanishes somewhere in } \overline{\Omega}, \quad u \not\equiv 0\}.$$

Moreover, there holds $\overline{\lambda} > \lambda_1$.

Proof. First, we are going to prove the strict inequality $\overline{\lambda} > \lambda_1$. Since for every $w \in L$ one has

$$\frac{\int_\Omega |\nabla w|^p dx + \beta\int_{\partial\Omega} |w|^p d\sigma}{\int_\Omega |w|^p dx} \le \max_{r\in\mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + w)|^p dx + \beta\int_{\partial\Omega} |r\varphi_1 + w|^p d\sigma}{\int_\Omega |r\varphi_1 + w|^p dx},$$

we conclude that

$$\lambda_1 \le \inf_{w\in L} \frac{\int_\Omega |\nabla w|^p dx + \beta\int_{\partial\Omega} |w|^p d\sigma}{\int_\Omega |w|^p dx} \le \overline{\lambda}. \tag{4.4}$$

Let us check that the first inequality in (4.4) is strict. On the contrary, we would find a sequence $(w_n) \subset L$ satisfying

$$\frac{\int_\Omega |\nabla w_n|^p dx + \beta\int_{\partial\Omega} |w_n|^p d\sigma}{\int_\Omega |w_n|^p dx} \to \lambda_1 \quad \text{as } n \to \infty.$$

Set $v_n = \frac{w_n}{\|w_n\|_\beta}$, where $\|\cdot\|_\beta$ denotes the equivalent norm on $W^{1,p}(\Omega)$ introduced in (4.3). We note that $\|v_n\|_\beta = 1$ and

$$\frac{1}{\int_\Omega |v_n|^p dx} \to \lambda_1 \quad \text{as } n \to \infty.$$

Due to the compact embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, there is a subsequence of (v_n), still denoted by (v_n), such that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$ and $v_n \to v$ uniformly on $\overline{\Omega}$. It follows that $v \in L$ and

$$\frac{\int_\Omega |\nabla v|^p dx + \beta \int_{\partial\Omega} |v|^p d\sigma}{\int_\Omega |v|^p dx} \le \lambda_1 = \frac{1}{\int_\Omega |v|^p dx},$$

which ensures that v is an eigenfunction in (1.3) corresponding to the first eigenvalue λ_1. This is a contradiction because every eigenfunction associated to λ_1 is strictly positive or negative on $\overline{\Omega}$, whereas $v \in L$. Hence, recalling (4.4), we get $\overline{\lambda} > \lambda_1$. Now we prove the first part in the theorem. We start by claiming that there exist $u \in L$ such that

$$\max_{r\in\mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u)|^p dx + \beta \int_{\partial\Omega} |r\varphi_1 + u|^p d\sigma}{\int_\Omega |r\varphi_1 + u|^p dx} = \overline{\lambda}. \tag{4.5}$$

By the definition of $\overline{\lambda}$, we can find sequences $(u_n) \subset L$ and $(r_n) \subset \mathbb{R}$ such that

$$\begin{aligned} &\max_{r\in\mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u_n)|^p dx + \beta \int_{\partial\Omega} |r\varphi_1 + u_n|^p d\sigma}{\int_\Omega |r\varphi_1 + u_n|^p dx} \\ &= \frac{\int_\Omega |\nabla(r_n\varphi_1 + u_n)|^p dx + \beta \int_{\partial\Omega} |r_n\varphi_1 + u_n|^p d\sigma}{\int_\Omega |r_n\varphi_1 + u_n|^p dx} \to \overline{\lambda} \quad \text{as } n \to \infty. \end{aligned} \tag{4.6}$$

Without loss of generality, we can assume that $\|u_n\|_{W^{1,p}(\Omega)} = 1$. The sequence (r_n) has to be bounded because otherwise there would exist a relabeled subsequence $r_n \to +\infty$, which results in

$$\frac{\int_\Omega |\nabla r_n\varphi_1 + u_n|^p dx + \beta \int_{\partial\Omega} |r_n\varphi_1 + u_n|^p d\sigma}{\int_\Omega |r_n\varphi_1 + u_n|^p dx} \to \lambda_1.$$

This implies that $\lambda_1 = \overline{\lambda}$, contradicting the inequality $\overline{\lambda} > \lambda_1$. Therefore, we may suppose that $r_n \to \widetilde{r} \in \mathbb{R}$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ as well as $u_n \to u$ uniformly in $\overline{\Omega}$, with some $u \in L$. Then, through (4.6), we see that (4.5) holds true.

To prove that $c(s) \to \overline{\lambda}$ as $s \to +\infty$, we argue by contradiction admitting that there exists $\delta > 0$ such that

$$\max_{t\in[-1,1]} \widetilde{J}_s(\gamma(t)) \ge \overline{\lambda} + \delta \quad \text{for all } \gamma \in \Gamma \text{ and all } s \ge 0.$$

Here the decreasing monotonicity of $c(s)$ has been used (see Proposition 4.2). Consider the path $\gamma \in \Gamma$ defined by

$$\gamma(t) = \frac{t\varphi_1 + (1-|t|)u}{\|t\varphi_1 + (1-|t|)u\|_{L^p(\Omega)}}, \quad t \in [-1,1],$$

with u given in (4.5). Proceeding as in the proof of Theorem 4.3, for every $s > 0$ we fix $t_s \in [-1,1]$ to satisfy

$$\max_{t\in[-1,1]} \widetilde{J}_s(\gamma(t)) = \widetilde{J}_s(\gamma(t_s))$$

and denote $v_s = t_s\varphi_1 + (1-|t_s|)u$. We have

$$\int_\Omega |\nabla v_s|^p dx + \beta \int_{\partial\Omega} |v_s|^p d\sigma - s\int_\Omega (v_s^+)^p dx \ge (\overline{\lambda} + \delta) \int_\Omega |v_s|^p dx. \tag{4.7}$$

From (4.7), we obtain $\int_\Omega (v_s^+)^p dx \to 0$ and $t_s \to \widetilde{t} \in [-1,1]$ as $s \to +\infty$, which yields $\widetilde{t}\varphi_1 \leq -(1-|\widetilde{t}|)u$. As $\varphi_1 > 0$ and u vanishes somewhere in $\overline{\Omega}$, we deduce that $\widetilde{t} \leq 0$. In addition, passing to the limit in (4.7) leads to

$$\begin{aligned} &\int_\Omega |\nabla(\widetilde{t}\varphi_1 + (1-|\widetilde{t}|)u)|^p dx + \beta \int_{\partial\Omega} |\widetilde{t}\varphi_1 + (1-|\widetilde{t}|)u|^p d\sigma \\ &\geq (\overline{\lambda} + \delta) \int_\Omega |\widetilde{t}\varphi_1 + (1-|\widetilde{t}|)u|^p dx. \end{aligned} \tag{4.8}$$

If $\widetilde{t} \neq -1$, (4.8) can be expressed as

$$\frac{\int_\Omega \left|\nabla\left(\frac{\widetilde{t}}{1+\widetilde{t}}\varphi_1 + u\right)\right|^p dx + \beta \int_{\partial\Omega} \left|\frac{\widetilde{t}}{1+\widetilde{t}}\varphi_1 + u\right|^p d\sigma}{\int_\Omega \left|\frac{\widetilde{t}}{1+\widetilde{t}}\varphi_1 + u\right|^p dx} \geq \overline{\lambda} + \delta.$$

Comparing with (4.5) reveals that a contradiction is reached. If $\widetilde{t} = -1$, in view of (4.8) and $\overline{\lambda} > \lambda_1$, we also arrive at a contradiction, which establishes the result. □

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