LOCAL $C^1(\overline{\Omega})$-MINIMIZERS VERSUS LOCAL $W^{1,p}(\Omega)$-MINIMIZERS OF NONSMOOTH FUNCTIONALS

PATRICK WINKERT

Abstract. We study not necessarily differentiable functionals of the form

$$J(u)=\frac{1}{p}\int_\Omega |\nabla u|^p dx+\frac{1}{p}\int_\Omega |u|^p dx+\int_\Omega j_1(x,u)dx+\int_{\partial\Omega} j_2(x,\gamma u)d\sigma$$

with $1<p<\infty$ involving locally Lipschitz functions $j_1:\Omega\times\mathbb{R}\to\mathbb{R}$ as well as $j_2:\partial\Omega\times\mathbb{R}\to\mathbb{R}$. We prove that local $C^1(\overline{\Omega})$-minimizers of J must be local $W^{1,p}(\Omega)$-minimizers of J.

1. Introduction

We consider the functional $J: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$J(u)=\frac{1}{p}\int_\Omega |\nabla u|^p dx+\frac{1}{p}\int_\Omega |u|^p dx+\int_\Omega j_1(x,u)dx+\int_{\partial\Omega} j_2(x,\gamma u)d\sigma$$

with $1<p<\infty$. The domain $\Omega\subset\mathbb{R}^N$ is supposed to be bounded with Lipschitz boundary $\partial\Omega$ and the nonlinearities $j_1:\Omega\times\mathbb{R}\to\mathbb{R}$ as well as $j_2:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ are measurable in the first argument and locally Lipschitz in the second one. By $\gamma: W^{1,p}(\Omega)\to L^{q_1}(\partial\Omega)$ for $1<q_1<p_*$ ($p_*=(N-1)p/(N-p)$ if $p<N$ and $p_*=+\infty$ if $p\geq N$), we denote the trace operator which is known to be linear, bounded and even compact. Note that $J: W^{1,p}(\Omega)\to\mathbb{R}$ does not have to be differentiable and that it corresponds to the following elliptic inclusion

$$-\Delta_p u+|u|^{p-2}u+\partial j_1(x,u)\ni 0 \qquad \text{in } \Omega,$$
$$\frac{\partial u}{\partial\nu}+\partial j_2(x,\gamma u)\ni 0 \qquad \text{on } \partial\Omega,$$

where $-\Delta_p u=-\operatorname{div}(|\nabla u|^{p-2}\nabla u), 1<p<\infty$, is the negative p-Laplacian. The symbol $\frac{\partial u}{\partial\nu}$ denotes the outward pointing conormal derivative associated with $-\Delta_p$ and $\partial j_k(x,u), k=1,2$, stands for Clarke's generalized gradient given by

$$\partial j_k(x,s)=\{\xi\in\mathbb{R}: j_k^o(x,s;r)\geq \xi r, \forall r\in\mathbb{R}\}.$$

The term $j_k^o(x,s;r)$ denotes the generalized directional derivative of the locally Lipschitz function $s\mapsto j_k(x,s)$ at s in the direction r defined by

$$j_k^o(x,s;r)=\limsup_{y\to s, t\downarrow 0}\frac{j_k(x,y+tr)-j_k(x,y)}{t},$$

(cf. [6, Chapter 2]). It is clear that $j_k^o(x,s;r)\in\mathbb{R}$ because $j_k(x,\cdot)$ is locally Lipschitz.

2010 *Mathematics Subject Classification.* 34A60, 35R70, 49J52, 49J53.
Key words and phrases. Clarke's generalized gradient, Local minimizers, Nonsmooth functionals, p-Laplacian.

The main goal of this paper is the comparison of local $C^1(\overline{\Omega})$ and local $W^{1,p}(\Omega)$-minimizers. That means that if $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of J, then u_0 is also a local $W^{1,p}(\Omega)$-minimizer of J. This result is stated in our main Theorem 3.1.

Such a result was first proven for functionals corresponding to elliptic equations with Dirichlet boundary values by Brezis and Nirenberg in [3] if $p = 2$. They consider potentials of the form

$$\Phi(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \int_\Omega F(x,u),$$

where $F(x,u) = \int_0^u f(x,s)ds$ with some Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$. An extension to the more general case $1 < p < \infty$ can be found in the paper of García Azorero et al. in [7]. We also refer the reader to [8] if $p > 2$. As regards nonsmooth functionals defined on $W_0^{1,p}(\Omega)$ with $2 \leq p < \infty$, we point to the paper [14]. A very inspiring paper about local minimizers of potentials associated with nonlinear parametric Neumann problems was published by Motreanu et al. in [13]. Therein, the authors study the functional

$$\phi_0(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F_0(z, x(z))dz, \quad \forall x \in W_n^{1,p}(\Omega)$$

with

$$W_n^{1,p}(\Omega) = \left\{ y \in W^{1,p}(\Omega) : \frac{\partial x}{\partial n} = 0 \right\},$$

where $\frac{\partial x}{\partial n}$ is the outer normal derivative of u and $F_0(z,x) = \int_0^x f_0(z,s)ds$, as well as $1 < p < \infty$. A similar result corresponding to nonsmooth functionals defined on $W_n^{1,p}(\Omega)$ for the case $2 \leq p < \infty$ was proved in [2]. We also refer the reader to the paper in [10] for $1 < p < \infty$.

A recent paper about the relationship between local $C^1(\overline{\Omega})$-minimizers and local $W^{1,p}(\Omega)$-minimizers of C^1-functionals has been treated by the author in [15]. The idea of the present paper was the generalization to the more general case of nonsmooth functionals defined on $W^{1,p}(\Omega)$ with $1 < p < \infty$ involving boundary integrals which in general do not vanish.

2. Hypotheses

We suppose the following conditions on the nonsmooth potentials $j_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $j_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$.

(H1) (i) $x \mapsto j_1(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$.
(ii) $s \mapsto j_1(x,s)$ is locally Lipschitz in $\mathbb{R}$ for almost all $x \in \Omega$.
(iii) There exists a constant $c_1 > 0$ such that for almost all $x \in \Omega$ and for all $\xi_1 \in \partial j_1(x,s)$ it holds that

$$|\xi_1| \leq c_1(1 + |s|^{q_0 - 1})$$

with $1 < q_0 < p^*$, where p^* is the Sobolev critical exponent

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

(H2) (i) $x \mapsto j_2(x,s)$ is measurable in $\partial\Omega$ for all $s \in \mathbb{R}$.
(ii) $s \mapsto j_2(x,s)$ is locally Lipschitz in $\mathbb{R}$ for almost all $x \in \partial\Omega$.

(iii) There exists a constant $c_2 > 0$ such that for almost all $x \in \partial\Omega$ and for all $\xi_2 \in \partial j_2(x, s)$ it holds that
$$|\xi_2| \leq c_2(1 + |s|^{q_1 - 1})$$
with $1 < q_1 < p_*$, where p_* is given by
$$p_* = \begin{cases} \dfrac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

(iv) Let $u \in W^{1,p}(\Omega)$. Then every $\xi_3 \in \partial j_2(x, u)$ satisfies the condition
$$|\xi_3(x_1) - \xi_3(x_2)| \leq L|x_1 - x_2|^\alpha,$$
for all x_1, x_2 in $\partial\Omega$ with $\alpha \in (0, 1]$.

Remark 2.1. *Note that the conditions above imply that the functional $J : W^{1,p}(\Omega) \to \mathbb{R}$ is locally Lipschitz (see* [4] *or* [9, p. 313]*). That guarantees, in particular, that Clarke's generalized gradient $s \mapsto \partial J(s)$ exists.*

3. $C^1(\overline{\Omega})$ versus $W^{1,p}(\Omega)$

Our main result is the following.

Theorem 3.1. *Let the conditions (H1) and (H2) be satisfied. If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of J, that is, there exists $r_1 > 0$ such that*
$$J(u_0) \leq J(u_0 + h) \quad \textit{for all } h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leq r_1,$$
then u_0 is a local minimizer of J in $W^{1,p}(\Omega)$, that is, there exists $r_2 > 0$ such that
$$J(u_0) \leq J(u_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\|_{W^{1,p}(\Omega)} \leq r_2.$$

Proof. Let $h \in C^1(\overline{\Omega})$ and let $\beta > 0$ small. Then we have
$$0 \leq \frac{J(u_0 + \beta h) - J(u_0)}{\beta},$$
which means that
$$0 \leq J^o(u_0; h) \quad \text{for all } h \in C^1(\overline{\Omega}).$$
The continuity of $J^o(u_0; \cdot)$ on $W^{1,p}(\Omega)$ and the density of $C^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$ imply
$$0 \leq J^o(u_0; h) \quad \text{for all } h \in W^{1,p}(\Omega).$$
Hence, we get
$$0 \in \partial J(u_0).$$
The inclusion above implies the existence of $h_1 \in L^{q_0'}(\Omega)$ with $h_1(x) \in \partial j_1(x, u_0(x))$ and $h_2 \in L^{q_1'}(\partial\Omega)$ with $h_2(x) \in \partial j_2(x, \gamma(u_0(x)))$ satisfying $1/q_0 + 1/q_0' = 1$ as well as $1/q_1 + 1/q_1' = 1$ such that
$$\begin{aligned} &\int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \nabla \varphi dx + \int_\Omega |u_0|^{p-2} u_0 \varphi dx \\ &+ \int_\Omega h_1 \varphi dx + \int_{\partial\Omega} h_2 \gamma \varphi d\sigma = 0, \ \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{3.1}$$

Note that equation (3.1) is the weak formulation of the Neumann boundary value problem

$$\begin{aligned} -\Delta_p u_0 &= -h_1 - |u_0|^{p-2}u_0 && \text{in } \Omega,\\ \frac{\partial u_0}{\partial \nu} &= -h_2 && \text{on } \partial\Omega, \end{aligned}$$

where $\frac{\partial u_0}{\partial \nu}$ means the outward pointing conormal and $-\Delta_p$ is the negative p-Laplacian. The regularity results in [16, Theorem 4.1 and Remark 2.2] along with [12, Theorem 2] ensure the existence of $\alpha \in (0,1)$ and $M > 0$ such that

$$u_0 \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_0\|_{C^{1,\alpha}(\overline{\Omega})} \le M. \tag{3.2}$$

In order to prove the theorem, we argue indirectly and suppose that the theorem is not valid. Hence, for any $\varepsilon > 0$ there exists $y_\varepsilon \in \overline{B_\varepsilon(u_0)}$ such that

$$J(y_\varepsilon) = \min\left\{ J(y) : y \in \overline{B_\varepsilon(u_0)} \right\} < J(u_0), \tag{3.3}$$

where $B_\varepsilon(u_0) = \{y \in W^{1,p}(\Omega) : \|y - u_0\|_{W^{1,p}(\Omega)} < \varepsilon\}$. More precisely, y_ε solves

$$\begin{cases} \min J(y) \\ y \in \overline{B_\varepsilon(u_0)}, F_\varepsilon(y) := \frac{1}{p}\left(\|y - u_0\|^p_{W^{1,p}(\Omega)} - \varepsilon^p \right) \le 0. \end{cases}$$

The usage of the nonsmooth multiplier rule of Clarke in [5, Theorem 1 and Proposition 13] yields the existence of a multiplier $\lambda_\varepsilon \ge 0$ such that

$$0 \in \partial J(y_\varepsilon) + \lambda_\varepsilon F'_\varepsilon(y_\varepsilon).$$

This means that we find $g_1 \in L^{q_0'}(\Omega)$ with $g_1(x) \in \partial j_1(x, y_\varepsilon(x))$ as well as $g_2 \in L^{q_1'}(\partial\Omega)$ with $g_2(x) \in \partial j_2(x, \gamma(y_\varepsilon(x)))$ to obtain

$$\begin{aligned} &\int_\Omega |\nabla y_\varepsilon|^{p-2} \nabla y_\varepsilon \nabla \varphi dx + \int_\Omega |y_\varepsilon|^{p-2} y_\varepsilon \varphi dx + \int_\Omega g_1 \varphi dx \\ &+ \int_{\partial\Omega} g_2 \gamma \varphi d\sigma + \lambda_\varepsilon \int_\Omega |\nabla (y_\varepsilon - u_0)|^{p-2} \nabla (y_\varepsilon - u_0) \nabla \varphi dx \\ &+ \lambda_\varepsilon \int_\Omega |y_\varepsilon - u_0|^{p-2} (y_\varepsilon - u_0) \varphi dx = 0, \end{aligned} \tag{3.4}$$

for all $\varphi \in W^{1,p}(\Omega)$. Next, we have to show that y_ε belongs to $L^\infty(\Omega)$ and hence to $C^{1,\alpha}(\overline{\Omega})$.
Case 1: $\lambda_\varepsilon = 0$ with $\varepsilon \in (0,1]$.
From (3.4) we see that y_ε solves the Neumann boundary value problem

$$\begin{aligned} -\Delta_p y_\varepsilon &= -g_1 - |y_\varepsilon|^{p-2} y_\varepsilon && \text{in } \Omega,\\ \frac{\partial y_\varepsilon}{\partial \nu} &= -g_2 && \text{on } \partial\Omega, \end{aligned}$$

As before, the regularity results in [16] and [12] yield (3.2) for y_ε.
Case 2: $0 < \lambda_\varepsilon \le 1$ with $\varepsilon \in (0,1]$.

Multiplying (3.1) with λ_ε and adding (3.4) yields

$$\begin{aligned}
&\int_\Omega |\nabla y_\varepsilon|^{p-2}\nabla y_\varepsilon \nabla\varphi dx + \lambda_\varepsilon \int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \nabla \varphi dx \\
&+ \lambda_\varepsilon \int_\Omega |\nabla(y_\varepsilon - u_0)|^{p-2}\nabla(y_\varepsilon - u_0)\nabla\varphi dx \\
&= -\int_\Omega (\lambda_\varepsilon h_1 + g_1 + \lambda_\varepsilon |u_0|^{p-2}u_0)\varphi dx \\
&\quad - \int_\Omega (\lambda_\varepsilon |y_\varepsilon - u_0|^{p-2}(y_\varepsilon - u_0) + |y_\varepsilon|^{p-2}y_\varepsilon)\varphi dx \\
&\quad - \int_{\partial\Omega} (\lambda_\varepsilon h_2 + g_2)\gamma\varphi d\sigma.
\end{aligned} \tag{3.5}$$

With (3.5) in mind, we introduce the operator $T_\varepsilon : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ given by

$$T_\varepsilon(x,\xi) = |\xi|^{p-2}\xi + \lambda_\varepsilon |H|^{p-2}H + \lambda_\varepsilon|\xi - H|^{p-2}(\xi - H),$$

where $H(x) = \nabla u_0(x)$ and $H \in (C^\alpha(\overline{\Omega}))^N$ for some $\alpha \in (0,1]$. It is clear that $T_\varepsilon(x,\xi) \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N)$. For $x \in \Omega$ we have

$$\begin{aligned}
&(T_\varepsilon(x,\xi),\xi)_{\mathbb{R}^N} \\
&= |\xi|^p + \lambda_\varepsilon(|\xi - H|^{p-2}(\xi - H) - |-H|^{p-2}(-H), \xi - H - (-H))_{\mathbb{R}^N} \\
&\geq |\xi|^p \quad \text{for all } \xi \in \mathbb{R}^N,
\end{aligned} \tag{3.6}$$

where $(\cdot,\cdot)_{\mathbb{R}^N}$ is the inner product in $\mathbb{R}^N$. The estimate (3.6) shows that T_ε satisfies a strong ellipticity condition. Hence, the equation in (3.5) is the weak formulation of the elliptic Neumann boundary value problem

$$\begin{aligned}
&-\operatorname{div} T_\varepsilon(x,\nabla y_\varepsilon) \\
&= -(\lambda_\varepsilon h_1 + g_1 + \lambda_\varepsilon(|u_0|^{p-2}u_0 + |y_\varepsilon - u_0|^{p-2}(y_\varepsilon - u_0)) + |y_\varepsilon|^{p-2}y_\varepsilon) \quad \text{in } \Omega, \\
&\frac{\partial v_\varepsilon}{\partial \nu} = -(\lambda_\varepsilon h_2 + g_2) \quad \text{on } \partial\Omega.
\end{aligned}$$

Using again the regularity results in [16] in combination with (3.6) and the growth conditions (H1)(iii) as well as (H2)(iii) proves $y_\varepsilon \in L^\infty(\Omega)$. Note that

$$|D_\xi T_\varepsilon(x,\xi)| \leq b_1 + b_2|\xi|^{p-2}, \tag{3.7}$$

where b_1, b_2 are some positive constants. We also obtain

$$\begin{aligned}
&(D_\xi T_\varepsilon(x,\xi)y,y)_{\mathbb{R}^N} \\
&= |\xi|^{p-2}|y|^2 + (p-2)|\xi|^{p-4}(\xi,y)^2_{\mathbb{R}^N} \\
&\quad + \lambda_\varepsilon|\xi - H|^{p-2}|y|^2 + \lambda_\varepsilon(p-2)|\xi - H|^{p-4}(\xi - H, y)^2_{\mathbb{R}^N} \\
&\geq \begin{cases} |\xi|^{p-2}|y|^2 & \text{if } p \geq 2 \\ (p-1)|\xi|^{p-2}|y|^2 & \text{if } 1 < p < 2 \end{cases} \\
&\geq \min\{1, p-1\}|\xi|^{p-2}|y|^2.
\end{aligned} \tag{3.8}$$

Because of (3.7) and (3.8), the assumptions of Lieberman in [12] are satisfied and thus, Theorem 2 in [12] ensures the existence of $\alpha \in (0,1)$ and $M > 0$, both independent of $\varepsilon \in (0,1]$, such that

$$y_\varepsilon \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|y_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq M, \quad \text{for all } \varepsilon \in (0,1]. \tag{3.9}$$

Case 3: $\lambda_\varepsilon > 1$ with $\varepsilon \in (0,1]$.
Multiplying (3.1) with -1, setting $v_\varepsilon = y_\varepsilon - u_0$ in (3.4) and adding these new equations yields

$$\begin{aligned}
&\int_\Omega |\nabla(u_0+v_\varepsilon)|^{p-2}\nabla(u_0+v_\varepsilon)\nabla\varphi dx - \int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \nabla\varphi dx \\
&+ \lambda_\varepsilon \int_\Omega |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla\varphi dx \\
&= \int_\Omega (|u_0|^{p-2}u_0 - |v_\varepsilon+u_0|^{p-2}(v_\varepsilon+u_0) - \lambda_\varepsilon |v_\varepsilon|^{p-2}v_\varepsilon)\varphi dx \\
&\quad + \int_\Omega (h_1-g_1)\varphi dx + \int_{\partial\Omega}(h_2-g_2)\gamma\varphi\sigma.
\end{aligned} \tag{3.10}$$

Defining again

$$T_\varepsilon(x,\xi) = \frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}(H+\xi) - |H|^{p-2}H) + |\xi|^{p-2}\xi$$

and rewriting (3.10) yields the equation

$$\begin{aligned}
&-\operatorname{div} T_\varepsilon(x,\nabla v_\varepsilon) \\
&= \frac{1}{\lambda_\varepsilon}(|u_0|^{p-2}u_0 - |v_\varepsilon+u_0|^{p-2}(v_\varepsilon+u_0) - \lambda_\varepsilon|v_\varepsilon|^{p-2}v_\varepsilon + h_1 - g_1) \quad && \text{in } \Omega, \\
&\frac{\partial v_\varepsilon}{\partial \nu} = \frac{1}{\lambda_\varepsilon}(h_2-g_2) && \text{on } \partial\Omega.
\end{aligned}$$

As above, we have the following estimates:

$$(T_\varepsilon(x,\xi),\xi)_{\mathbb{R}^N} \geq |\xi|^p \quad \text{for all } \xi \in \mathbb{R}^N, \tag{3.11}$$

$$|D_\xi T_\varepsilon(x,\xi)| \leq a_1 + a_2|\xi|^{p-2}, \tag{3.12}$$

$$(D_\xi T_\varepsilon(x,\xi)y,y)_{\mathbb{R}^N} \geq \min\{1,p-1\}|\xi|^{p-2}|y|^2, \tag{3.13}$$

with some positive constants a_1, a_2. Due to (3.11) along with [16], we obtain $v_\varepsilon \in L^\infty(\Omega)$. The statements (3.12) as well as (3.13) allow us to apply again the regularity results of Lieberman which implies the existence of $\alpha \in (0,1)$ and $M > 0$, both independent of $\varepsilon \in (0,1]$, such that (3.9) holds for v_ε. Because of $y_\varepsilon = v_\varepsilon + u_0$ and (3.2), we obtain (3.9) in the case $\lambda_\varepsilon > 1$. Summarizing, we have proved that $y_\varepsilon \in L^\infty(\Omega)$ and $y_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$ for all $\varepsilon \in (0,1]$ with $\alpha \in (0,1)$.
Let $\varepsilon \downarrow 0$. We know that the embedding $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ is compact (cf. [11, p. 38] or [1, p. 11]). Hence, we find a subsequence y_{ε_n} of y_ε such that $y_{\varepsilon_n} \to \widetilde{y}$ in $C^1(\overline{\Omega})$. By construction we have $y_{\varepsilon_n} \to u_0$ in $W^{1,p}(\Omega)$ which yields $\widetilde{y} = u_0$. So, for n sufficiently large, say $n \geq n_0$, we have

$$\|y_{\varepsilon_n} - u_0\|_{C^1(\overline{\Omega})} \leq r_1,$$

which provides

$$J(u_0) \leq J(y_{\varepsilon_n}). \tag{3.14}$$

However, the choice of the sequence (y_{ε_n}) implies

$$J(y_{\varepsilon_n}) < J(u_0), \ \forall n \geq n_0$$

(see (3.3)) which is a contradiction to (3.14). This completes the proof of the theorem. □

References

[1] R. A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] G. Barletta and N. S. Papageorgiou. A multiplicity theorem for the Neumann p-Laplacian with an asymmetric nonsmooth potential. *J. Global Optim.*, 39(3):365–392, 2007.

[3] H. Brezis and L. Nirenberg. H^1 versus C^1 local minimizers. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(5):465–472, 1993.

[4] K.C. Chang. Variational methods for nondifferentiable functionals and their applications to partial differential equations. *J. Math. Anal. Appl.*, 80(1):102–129, 1981.

[5] F. H. Clarke. A new approach to Lagrange multipliers. *Math. Oper. Res.*, 1(2):165–174, 1976.

[6] F. H. Clarke. *Optimization and nonsmooth analysis*, volume 5 of *Classics in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1990.

[7] J. P. García Azorero, I. Peral Alonso, and J. J. Manfredi. Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations. *Commun. Contemp. Math.*, 2(3):385–404, 2000.

[8] Z. Guo and Z. Zhang. $W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations. *J. Math. Anal. Appl.*, 286(1):32–50, 2003.

[9] S. Hu and N. S. Papageorgiou. *Handbook of multivalued analysis. Vol. II*, volume 500 of *Mathematics and its Applications.* Kluwer Academic Publishers, Dordrecht, 2000.

[10] A. Iannizzotto and N. S. Papageorgiou. Existence of three nontrivial solutions for nonlinear Neumann hemivariational inequalities. *Nonlinear Anal.*, 70(9):3285–3297, 2009.

[11] A. Kufner, O. John, and S. Fučík. *Function spaces.* Noordhoff International Publishing, Leyden, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

[12] G. M. Lieberman. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.*, 12(11):1203–1219, 1988.

[13] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou. Nonlinear neumann problems near resonance. *Indiana Univ. Math. J.*, 58:1257–1280, 2008.

[14] D. Motreanu and N. S. Papageorgiou. Multiple solutions for nonlinear elliptic equations at resonance with a nonsmooth potential. *Nonlinear Anal.*, 56(8):1211–1234, 2004.

[15] P. Winkert. Constant-sign and sign-changing solutions for nonlinear elliptic equations with neumann boundary values. *Adv. Differential Equations.* (in press).

[16] P. Winkert. L^∞-estimates for nonlinear elliptic neumann boundary value problems. *NoDEA Nonlinear Differential Equations Appl.* doi: 10.1007/s00030-009-0054-5.

Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany

E-mail address: `winkert@math.tu-berlin.de`