T-SYMMETRICAL TENSOR DIFFERENTIAL FORMS WITH LOGARITHMIC POLES ALONG A HYPERSURFACE SECTION

PETER BRÜCKMANN AND PATRICK WINKERT

ABSTRACT. The aim of this paper is to investigate T-symmetrical tensor differential forms with logarithmic poles on the projective space $\mathbb{P}^N$ and on complete intersections $Y \subset \mathbb{P}^N$. Let $H \subset \mathbb{P}^N, N \geq 2$, be a nonsingular irreducible algebraic hypersurface which implies that $D = H$ is a prime divisor in $\mathbb{P}^N$. The main goal of this paper is the study of the locally free sheaves $\Omega^T_{\mathbb{P}^N}(\log D)$ and the calculation of their cohomology groups. In addition, for complete intersections $Y \subset \mathbb{P}^N$ we give some vanishing theorems and recursion formulas.

1. Introduction

The symmetry properties of tensors are important in physics and certain areas of mathematics. In the following, let $k$ be the ground field which is assumed to be algebraically closed satisfying $\text{char}(k) = 0$. We denote by $H \subset \mathbb{P}^N_k, N \geq 2$, a nonsingular, irreducible, algebraic hypersurface defined by the equation $F = 0$, where $\deg F = m$. Then $D = H$ gives a prime divisor of degree $m$ in $\mathbb{P}^N_k$. The aim of this paper is the calculation of the dimension of the cohomology groups $H^q(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t))$ with general twist $t \in \mathbb{Z}$, where $T$ is a Young tableau specified later. $\Omega^T_{\mathbb{P}^N}(\log D)$ denotes the so-called sheaf of germs of $T$-symmetrical tensor differential forms with logarithmic poles along the prime divisor $D$ (cf. [4], [7], [3]). In addition, we consider the associated cohomology groups of nonsingular, irreducible, $n$-dimensional complete intersections $Y \subset \mathbb{P}^N, n \geq 2$. In this case, let the prime divisor $D = Y \cap H$ be the intersection of $Y$ and the hypersurface $H$. As special cases, we investigate the alternating and the symmetric differential forms on $\mathbb{P}^N$ and on $Y$, respectively.

2. Notations and Preliminaries

Let $\Omega^1_X$ be the sheaf of germs of regular algebraic differential forms on a $n$-dimensional nonsingular, projective variety $X \subseteq \mathbb{P}^N$ and let $\Omega^r_X = \wedge^r \Omega^1_X$ and $S^r \Omega^1_X$ be the sheaves of alternating and symmetric differential forms on $X$, alternatively. We denote by $(\Omega^1_X)^\otimes r$ the $r$-th tensor power of $\Omega^1_X$. The coherent sheaves $\Omega^r_X, \Omega^r_X, S^r \Omega^1_X$ and $(\Omega^1_X)^\otimes r$ are locally free on $X$ with the rank $n, \binom{n}{r}, \binom{n+r-1}{r}$ and $n^r$, respectively.

The irreducible representations of the symmetric group $S_r$ correspond to the conjugacy classes of $S_r$. These are given by partitions $(l) : r = l_1 + \ldots + l_d$ with $l_i \in \mathbb{Z}, l_1 \geq l_2 \geq \ldots \geq l_d \geq 1$. Partition $(l)$ can be described by a so-called Young diagram $T$ with $r$ boxes and the row lengths $l_1, \ldots, l_d$. The column lengths of $T$ will be denoted by $d_1, \ldots, d_l$ and we set $d = d_1 = \text{depth} T$ and

2000 Mathematics Subject Classification. 14F10, 14M10, 14F17, 55N30.
Key words and phrases. Young tableaux, complete intersections, algebraic differential forms.
l = l_t = \text{length } T, \text{ respectively. Clearly, } d_1 \geq d_2 \geq \ldots \geq d_l \geq 1 \text{ and the equations } 
\sum_{j=1}^l d_j = \sum_{i=1}^d l_i = r \text{ are fulfilled. Moreover, we put } l_i = 0 \text{ for } i > d \text{ and } d_j = 0 \text{ for } j > l. \text{ The "hook-length" of the box inside the } i-\text{th row and the } j-\text{th column of the Young diagram is defined by } h_{i,j} = l_i - i + d_j - j + 1 \text{ and the degree of the associated irreducible representation is equal to }
\nu(t) = \frac{r!}{\prod h_{i,j}} = \frac{r!}{d!} \cdot \frac{d!}{\prod_{i=1}^d (l_i + d - i)!} \cdot \prod_{1 \leq i < j \leq d} \left( \frac{l_i - l_j}{j - i} + 1 \right) = r! \cdot \det\left(\frac{1}{\Gamma(l_i + 1 - i + j)}\right)_{i,j=1,...,d} \quad (\text{cf. } \cite{5}).

A numbering of the } r \text{ boxes of a given Young diagram by the integers } 1, 2, \ldots, r \text{ in any order is said to be a Young tableau which for simplicity again will be denoted by } T. \text{ Now, one has an idempotent } e_T \text{ in the group algebra } k \cdot S_r \text{ defined by } 
\nu(T) = \sum_{q \in Q_T} \text{sgn}(q) \cdot q \circ \left( \sum_{p \in P_T} p \right),
\text{ where the subgroups } P_T \text{ and } Q_T \text{ of } S_r \text{ are given as follows: } P_T = \{ p \in S_r : p \text{ preserves each row of } T \}, \text{ } Q_T = \{ q \in S_r : q \text{ preserves each column of } T \}.

The idempotent } e_T \text{ is called Young symmetrizer (cf. } \cite{5}). \text{ If the numbering of the boxes of the Young tableau generates inside every row and every column monotone increasing sequences, we speak of a standard tableau. The number of all standard tableaux to all Young diagrams with } r \text{ boxes is equal to } \nu(T). \text{ We denote by } D(r) \text{ the set of all standard tableaux to all Young diagrams with } r \text{ boxes.}

For a variety } X, \text{ the notation } \Omega_X^{\otimes r} = (\Omega_X^1)^{\otimes r} \text{ stands for the sheaf of germs of regular algebraic tensor differential forms. This implies that the symmetric group } S_r \text{ and the related group algebra } k \cdot S_r \text{ act on } \Omega_X^{\otimes r} \text{ defined by } p(a_1 \otimes \ldots \otimes a_r) = a_{p^{-1}(1)} \otimes \ldots \otimes a_{p^{-1}(r)} \text{ for all } p \in S_r. \text{ That means, mapping } p \text{ permutes the spots inside the tensor product. Furthermore, it holds } 
\Omega_X^{\otimes r} = \bigoplus_{T \in D(r)} \Omega_X^T
\text{ with } \Omega_X^T = e_T(\Omega_X^{\otimes r}), \text{ where } \Omega_X^T \text{ is called the sheaf of germs of } T\text{-symmetrical tensor differential forms or simply the } T\text{-power of } \Omega_X^T. \text{ If two Young tableaux } T \text{ and } \tilde{T} \text{ possess the same Young diagram, we have } \Omega_X^T \cong \Omega_X^{\tilde{T}}.

Under the assumption } \text{depth } T \leq \dim X \text{ with a smooth } n\text{-dimensional variety } X \text{ the belonging sheaf } \Omega_X^T \text{ is locally free of rank }
\prod_{1 \leq i < j \leq n} \left( \frac{l_i - l_j}{j - i} + 1 \right) = \left( \prod_{i=1}^{n-1} i! \right)^{-1} \cdot \Delta(l_1 - 1, l_2 - 2, \ldots , l_n - n),
\text{ where } \Delta(t_1, t_2, \ldots , t_n) = \prod_{1 \leq i < j \leq n} (t_i - t_j) \text{ denotes the Vandermonde determinant. If depth } T > \dim X \text{ then we have } \Omega_X^T = 0. \text{ In the special cases } \Omega_X^1 = \wedge^r \Omega_X^1 \text{ and } S^r \Omega_X^1 \text{ the Young tableau has only one column and one row, respectively. In the same way the } T\text{-power } F^T \text{ of an arbitrary coherent algebraic sheaf } F \text{ is defined. One has for instance } \Omega_X^T(\log D) = (\Omega_X^1(\log D))^T.

Furthermore, we describe the } T\text{-power of an algebraic complex (cf. } \cite{3}): \text{ Let } R \text{ be a commutative ring which contains the algebraically closed ground field } k
fulfilling \( \text{char}(k) = 0 \). We consider an algebraic complex \( K \) of \( R \)-modules given by \( K : K_0 \xrightarrow{d} K_1 \xrightarrow{d} K_2 \xrightarrow{d} \ldots \) with \( d^2 = 0 \). Then the \( r \)-th tensor power \( P = K^\otimes r \) of \( K \) is defined by \( P = K^\otimes r : P_0 \xrightarrow{\delta} P_1 \xrightarrow{\delta} P_2 \xrightarrow{\delta} \ldots \) with \( P_0 = \bigoplus_{s_1 + \ldots + s_r = s} K_{s_1} \otimes \ldots \otimes K_{s_r} \) and \( \delta(b_1 \otimes \ldots \otimes b_r) = \sum_{i=1}^r (-1)^{s_1 + \ldots + s_{i-1}} b_1 \otimes \ldots \otimes b_{i-1} \otimes d(b_i) \otimes b_{i+1} \otimes \ldots \otimes b_r \), where \( b_j \in K_{s_j} \) for all \( j \). Again the symmetric group \( S_r \) acts on this tensor power by permutation of the spots inside the tensor product.

In order to obtain such an action of \( S_r \) on \( P = K^\otimes r \), which commutes with \( \delta \), we introduce additionally a sign as follows:

1. \( \sigma(p ; s_1, \ldots, s_r) := \sum_{\pi(n)} s_{\pi(1)} \cdot s_{\pi(2)} \) for all \( p \in S_r \)
2. \( p(b_1 \otimes \ldots \otimes b_r) := (-1)^{\sigma(p ; s_1, \ldots, s_r)} \cdot b_{p(1)} \otimes \ldots \otimes b_{p(r)} \)

Then one has

\[
P_s = \bigoplus_{T \in D(r)} K_s(T), \quad K^\otimes r = \bigoplus_{T \in D(r)} K(T), \quad H^* (K^\otimes r) = \bigoplus_{T \in D(r)} H^*(K(T))
\]

with \( K_s(T) = e_T(P_s) \) and \( K(T) = e_T(K^\otimes r) : K_0(T) \xrightarrow{\delta} K_1(T) \xrightarrow{\delta} K_2(T) \xrightarrow{\delta} \ldots \). This complex \( K(T) \) is said to be the \( T \)-power of \( K \). If two Young tableaux \( T \) and \( \tilde{T} \) possess the same Young diagram, one has \( K(T) \cong K(\tilde{T}) \). For an exact sequence \( K \) the \( T \)-power \( K(T) \) of \( K \) is also an exact sequence.

Now, let \( X \subseteq \mathbb{P}^N \) be a projective variety satisfying \( \omega_X \cong \mathcal{O}_X(n_X) \) for some \( n_X \in \mathbb{Z} \), where \( \omega_X \) stands for the canonical line bundle. This implies under the assumptions \( d = \text{depth} T = \dim X = n \) and \( l = \text{length} T > 1 \) the isomorphism

\[
\Omega_X^T \cong \Omega_X^{T'} \otimes \omega_X \cong \Omega_X^{T'}(n_X),
\]

where \( T' \) arises from \( T \) by deleting the first column of \( T \). In the case \( d = \text{depth} T = \dim X = n \) and \( l = \text{length} T = 1 \) (i.e. \( T \) has only one column) we have the isomorphism \( \Omega_X^T \cong \Omega_X^T \cong \omega_X \cong \mathcal{O}_X(n_X) \).

An important tool in our considerations will be the Serre duality: Suppose the Young tableau \( T \) has the column lengths \( d_1, \ldots, d_l \) satisfying \( d_1 = d = \text{depth} T \leq \dim X = n \). We get an associated Young tableau \( T^* \) by the column lengths \( d_i^* = n - d_{i+1-j} \) for all \( j = 1, \ldots, l \). One verifies readily that in case \( \text{depth} T < n \) holds \( (T^*)^* = T \).

The next lemma delivers some duality relations about the dimensions of cohomology groups.

**Lemma 2.1.** Let \( Y = H_1 \cap \ldots \cap H_{N-n} \subseteq \mathbb{P}^N \) be a \( n \)-dimensional, nonsingular, irreducible, complete intersection defined by algebraic hypersurfaces \( H_i \subseteq \mathbb{P}^N \) satisfying \( \text{deg} F_i = m_i \). The dimension of \( Y \) is \( n \). In this case, let the prime divisor \( D = Y \cap H \) be the intersection of \( Y \) and hypersurface \( H : F = 0 \) of degree \( m \). Assume that \( D \) also becomes a nonsingular irreducible complete intersection of dimension \( n - 1 \). Then one has:

- (i) \( \dim H^q(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^r(\log D)(t)) = \dim H^{N-q}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{N-r}(\log D)(-t-m)) \)
- (ii) \( \dim H^q(Y, \Omega_X^T(\log D)(t)) = \dim H^{n-q}(Y, \Omega_X^{N-r}(\log D)(-t-m)) \)
- (iii) \( \dim H^q(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^r(\log D)(t)) = \dim H^{N-q}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^{N-r}(\log D)(-t-l \cdot m + (l-1)(N+1))) \)
- (iv) \( \dim H^q(Y, \Omega_X^{T^*}(\log D)(t)) = \dim H^{n-q}(Y, \Omega_X^{N-r}(\log D)(-t-l \cdot m - (l-1)(\sum_{i=1}^{N-n} m_i - N-1))) \)
\( \text{(v) dim } H^q(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (log \, D)(t)) \\
= \text{dim } H^{N-q}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^1 (log \, D)(-t - r \cdot m + (r - 1)(N + 1))) \)

where \( T^* \) denotes a rectangle with \( N - 1 \) rows and \( r \) columns.

\( \text{(vi) dim } H^q(Y, S^r \Omega_{\mathbb{P}^N}^1 (log \, D)(t)) \\
= \text{dim } H^{n-q}(Y, \Omega_{\mathbb{P}^N}^1 (log \, D)(-t - r \cdot m - (r - 1)(\sum_{i=1}^{N-n} m_i - N - 1))) \)

where \( T^* \) denotes a rectangle with \( n - 1 \) rows and \( r \) columns.

**Proof.** We consider the following exact sequence (cf. [4])
\[
0 \longrightarrow \Omega_{\mathbb{P}^N}^1 (log \, D)(-m) \longrightarrow \Omega_{\mathbb{P}^N}^1 \longrightarrow \Omega_{\mathbb{P}^N}^1 (log \, D)(m) \longrightarrow 0.
\]

For \( r = N \) we have \( \Omega_{\mathbb{P}^N}^1 (log \, D)(m) = 0 \), i.e. \( \Omega_{\mathbb{P}^N}^1 (log \, D)(m) \cong (m, m - N - 1) \), which means the vector space \( H^0(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^1 (log \, D)(t)) \) is dual to \( H^{N-q}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^1 (log \, D)(-t - m + N + 1)) \). Setting \( \Omega_{\mathbb{P}^N}^1 (log \, D)(m) \cong \mathcal{O}_{\mathbb{P}^N}(m + \sum_{i=1}^{N-n} m_i - N - 1) \). Now, let \( T \) be a Young tableau with \( r \) boxes, given by the row lengths \( l_1, \ldots, l_r \) and the column lengths \( d_1, \ldots, d_l \) where \( d = d_1 = \text{depth} \, T \) and \( l = l_1 = \text{length} \, T \). The Young tableau \( T^* \) has the column lengths \( d_j = n - d_{i+1} - j \) for all \( j \in \{1, \ldots, l\} \) and we have again \( \Omega_{\mathbb{P}^N}^1 (log \, D)(m) \cong \mathcal{O}_{\mathbb{P}^N}(m - N - 1) \). From the pairing \( \Omega_{\mathbb{P}^N}^1 (log \, D)(t) \times \Omega_{\mathbb{P}^N}^1 (log \, D)(t) \cong \mathcal{O}_{\mathbb{P}^N}(m - l \cdot (m - N - 1)) \rightarrow \mathcal{O}_{\mathbb{P}^N} \) follows
\[
\text{Hom}(\Omega_{\mathbb{P}^N}^1 (log \, D)(t), \mathcal{O}_{\mathbb{P}^N}) \cong \Omega_{\mathbb{P}^N}^1 (log \, D)(-t - l \cdot (m - N - 1)),
\]
which shows assertion (iii). In order to show the formula for complete intersections \( Y \) instead of \( \mathbb{P}^N \), we replace \( -N - 1 \) by \( \sum_{i=1}^{N-n} m_i - N - 1 \). Choosing \( l = r \) (depth \( T = 1 \)) in (iii) and (iv) proves (v) and (vi), respectively. \( \Box \)

For a projective variety \( X \subseteq \mathbb{P}^N \) and a coherent sheaf \( \mathcal{F} \) on \( X \) the dimensions \( \dim_k H^q(X, \mathcal{F}) \) are finite and we have the so-called Euler-Poincaré characteristic given by
\[
\chi(X, \mathcal{F}) = \sum_{q=0}^{\dim_k X} (-1)^q \cdot \dim H^q(X, \mathcal{F}).
\]

From a short exact sequence
\[
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0
\]
with coherent sheaves \( \mathcal{F}, \mathcal{G}, \mathcal{H} \) on \( X \) we obtain the equation \( \chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H}) \). Under the assumptions above, we also know, that for a short exact sequence of coherent sheaves on \( X \) there exists a long exact sequence for the associated cohomology groups. For every coherent sheaf \( \mathcal{F} \) on the projective variety \( X \subseteq \mathbb{P}^N \) there exists a polynomial \( P(X, \mathcal{F})(t) \in \mathbb{Q}[t] \) of degree dim \( X \) which fulfills \( \chi(X, \mathcal{F})(t) = P(X, \mathcal{F})(t) \) for all \( t \in \mathbb{Z} \). \( P(X, \mathcal{F})(t) \) is said to be the Hilbert polynomial of \( \mathcal{F} \) (cf. [8], [6], [7]). For example, the structure sheaf on \( \mathbb{P}^N \) has the following Hilbert polynomial
\[
P(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N})(t) = \chi(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N})(t) = \frac{(t + N) \cdot \ldots \cdot (t + 1)}{N!}.
\]

3. The Projective Space \( \mathbb{P}^N \)

In the following, we change the meaning of the binomial coefficient setting \( \binom{\alpha}{\beta} = 0 \) for all \( \alpha \in \mathbb{Z}, \beta \in \mathbb{N} \) satisfying \( \alpha < \beta \), in particular: \( \binom{n}{0} = 0 \) if \( \alpha < 0 \). For instance:
\[
\text{dim } H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = \binom{+N}{t}, \text{ dim } H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = \binom{-1}{t}, \text{ dim } H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = 0 \text{ for } 0 < q < N.
\]

Let \( H \subseteq \mathbb{P}^N \) be a nonsingular, irreducible, algebraic hypersurface defined by the equation \( F = 0 \), that means, \( D = H \) is a prime divisor in \( \mathbb{P}^N \). Both \( F \) and \( D \) are of degree \( m \) and \( D = H \) has dimension \( N - 1 \).
3.1. Alternating Differential Forms. We denote by $\Omega^r_{\mathbb{P}^N}$ the local free sheaf of germs of alternating differential forms on the projective space $\mathbb{P}^N$ and consider the following sequence $(t \in \mathbb{Z})$

$$0 \to \Omega^r_{\mathbb{P}^N}(t) \to \Omega^r_{\mathbb{P}^N}(\log D)(t) \to \Omega^r_D(t) \to 0,$$

which is known to be exact (cf. [4]). The dimensions of the cohomology groups $H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(t))$ and $H^q(D, \Omega^r_D(t))$ are calculated in [1], where we also find the following exact sequences

$$0 \to \Omega^r_{\mathbb{P}^N}(t-m) \to \Omega^r_{\mathbb{P}^N}(t) \xrightarrow{\alpha} \mathcal{O}_D(t) \otimes \Omega^r_{\mathbb{P}^N} \to 0,$$

$$0 \to \Omega^r_D(t-m) \to \mathcal{O}_D(t) \otimes \Omega^r_{\mathbb{P}^N} \xrightarrow{\beta} \Omega^r_D(t) \to 0.$$

The mapping $\varphi^* := \beta \circ \alpha$ means the restriction of the differential forms on $\mathbb{P}^N$ to the hypersurface $D = H$. In the case $r = 1$, one has to replace the sheaf $\Omega^r_D$ by the structure sheaf $\mathcal{O}_D$. For $0 < q < N$ we have $\dim H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(t)) = \delta_{q,r} \cdot \delta_{t,0}$ (Kronecker-$\delta$) and we know by [1, Lemma 4] a base element of $H^r(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N})$ which is given by the cohomology class of the cocycle $\omega^{(r)} \in C^r(U, \Omega^r_{\mathbb{P}^N})$ defined by

$$\omega^{(r)}_{i_0, \ldots, i_r} = \frac{x_{i_0}}{x_{i_r}} \cdot d\frac{x_{i_1}}{x_{i_0}} \wedge d\frac{x_{i_2}}{x_{i_1}} \wedge \ldots \wedge d\frac{x_{i_r}}{x_{i_{r-1}}},$$

(3.2)

$U$ stands for the affine open covering of $\mathbb{P}^N$ by the affine spaces $U_i = \{x_i \neq 0\}$. For $r = 1$, in particular, $\omega^{(1)}_{i_0, i_1} = \frac{x_{i_0}}{x_{i_1}} \cdot d\frac{x_{i_1}}{x_{i_0}}$ is a logarithmic differential. We may represent (3.2) by

$$\omega^{(r)}_{i_0, \ldots, i_r} = \omega^{(1)}_{i_0, i_1} \wedge \omega^{(1)}_{i_1, i_2} \wedge \ldots \wedge \omega^{(1)}_{i_{r-1}, i_r},$$

which is an outer product of logarithmic differential forms. In the case $q = r = N$, $t = 0$ the cochain $\omega^{(N)}$ creates a base of $H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N})$ (cf. [1, Lemma 2]). Finally, we set $\omega^{(0)} = 1$.

Lemma 3.1. Let $0 < r \leq N$. Then the homomorphism $d : H^{r-1}(D, \Omega^r_D) \to H^r(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N})$ in the long homology sequence with respect to the exact sequence

$$0 \to \Omega^r_{\mathbb{P}^N} \to \Omega^r_{\mathbb{P}^N}(\log D) \to \Omega^r_D \to 0$$

is epimorphic. If in addition $2(r - 1) \neq N - 1$ is valid, then $d$ is an isomorphism.

Proof. We calculate the image of the cohomology class of $\omega^{(r-1)}$ at the composition

$$H^{r-1}(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}) \xrightarrow{\varphi^*} H^{r-1}(D, \Omega^r_D) \xrightarrow{d} H^r(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N})$$

denote $\varphi^*(\omega^{(r-1)})$ again by $\omega^{(r-1)}$. Let $U$ be the affine, open covering of $\mathbb{P}^N$ given by the affine spaces $U_i = \{x_i \neq 0\}$. We consider the following commutative diagram

$$
\begin{array}{cccccc}
0 & \to & C^{r-1}(U, \Omega^r_{\mathbb{P}^N}) & \to & C^{r-1}(U, \Omega^r_{\mathbb{P}^N}(\log D)) & \to & C^{r-1}(U, \Omega^r_D) & \to & 0 \\
0 & \to & C^r(U, \Omega^r_{\mathbb{P}^N}) & \to & C^r(U, \Omega^r_{\mathbb{P}^N}(\log D)) & \to & C^r(U, \Omega^r_D) & \to & 0 \\
\end{array}
$$

where the cocycle $\omega^{(r-1)} \in C^{r-1}(U, \Omega^r_D)$ possesses in $C^{r-1}(U, \Omega^r_{\mathbb{P}^N}(\log D))$ the preimage $\varphi$ defined by $\varphi_{i_0, \ldots, i_r} = \omega^{(r-1)}_{i_0, \ldots, i_r} \wedge \frac{x_{i_0}}{x_{i_r}} \cdot d\frac{x_{i_r}}{x_{i_{r-1}}}$ (cf. [4]). Elementary calculations show that $d\omega^{(r-1)} = (-1)^r \cdot m \cdot \omega^{(r)} \in C^r(U, \Omega^r_{\mathbb{P}^N})$. Therefore, the cocycle
$d\omega^{(r-1)} \in \mathcal{C}^r(\Omega, \Omega_{p,n}^r)$ is nonzero and the associated cohomology class is a base of $H^r(\mathbb{P}^N, \Omega_{p,n}^r)$. Thus, the homomorphism $d : H^{r-1}(\mathbb{P}^N, \Omega_{p,n}^{r-1}) \to H^r(\mathbb{P}^N, \Omega_{p,n}^r)$ is epimorphic. In the case $2(r-1) \neq N-1$, we obtain $\dim H^{r-1}(\mathbb{P}^N, \Omega_{p,n}^{r-1}) = 1$ by [1, Satz 2 and Lemma 5], which implies that $d$ is an isomorphism.

**Theorem 3.2.**

Let $D \subset \mathbb{P}^N$ be a smooth algebraic hypersurface of degree $m$ ($N \geq 2$).

(a) For each $r \in \{1, \ldots, N-1\}$ one has:

$$\dim H^0(\mathbb{P}^N, \Omega_{p,n}^r \log D)(t)) = \sum_{i=0}^r (-1)^i \cdot \binom{N+1}{i} \cdot \binom{t + N - i \cdot (m-1) - r}{N}.$$

(b) For all $r \in \{1, \ldots, N-1\}$ holds: $H^0(\mathbb{P}^N, \Omega_{p,n}^r \log D)(t)) \neq 0 \iff t \geq r$.

(c) In the case $r = N$ one has: $\dim H^0(\mathbb{P}^N, \Omega_{p,n}^N \log D)(t)) = \binom{t + m - 1}{N}$.

(d) If $D \subset \mathbb{P}^N$ is a hyperplane ($m = 1$), then it holds:

$$\dim H^0(\mathbb{P}^N, \Omega_{p,n}^r \log D)(t)) = \binom{N}{r} \cdot \binom{t + N - r}{N}.$$

**Proof.** The formula (a) follows directly from the long exact cohomology sequence related to the exact sequence in (3.1) by applying Lemma 3.1. For $r = N$ we obtain $\Omega_{p,n}^N \log D \cong \Omega_{p,n}^N \log D \cong \mathcal{O}_{p,n}(m - N - 1)$ which yields (c). (a) obviously implies (b) and (d).

**Theorem 3.3.**

(a) Let $0 < q < N$, $q + r \neq N$ and $r \geq 1$.

Then we obtain $H^q(\mathbb{P}^N, \Omega_{p,n}^r \log D)(t)) = 0$ for all $t \in \mathbb{Z}$.

(b) For $1 \leq r \leq N-1$ it follows:

$$\dim H^{N-r}(\mathbb{P}^N, \Omega_{p,n}^r \log D)(t)) = \sum_{i=0}^{N+1} (-1)^i \cdot \binom{N+1}{i} \cdot \binom{t + N - i \cdot (m-1)}{N}.$$

That means: If $D$ is a hyperplane ($m = 1$), then we have $H^{N-r}(\mathbb{P}^N, \Omega_{p,n}^r \log D)(t)) = 0$ for all $t \in \mathbb{Z}$.

(c) For $1 \leq r \leq N-1$ one has:

$$\dim H^N(\mathbb{P}^N, \Omega_{p,n}^r \log D)(t)) = \sum_{i=0}^{N-r} (-1)^i \cdot \binom{N+1}{N-r-i} \cdot \binom{t - m - i \cdot (m-1) + r}{N}.$$

If $D$ is a hyperplane ($m = 1$), then we get:

$$\dim H^N(\mathbb{P}^N, \Omega_{p,n}^r \log D)(t)) = \binom{N}{r} \cdot \binom{-t - 1 + r}{N}.$$
Proof. We consider the following exact sequence
\[
\ldots \longrightarrow H^{q-1}(D, \Omega_{D}^{-1}(t)) \xrightarrow{d_1} H^q(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(t)) \longrightarrow \]
\[
H^q(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(\log D)(t)) \longrightarrow H^q(D, \Omega_{D}^{-1}(t)) \xrightarrow{d_2} H^{q+1}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(t)) \longrightarrow \ldots ,
\]
and assume 0 < q , 0 < r and q + r < N. By Lemma 3.1 the mappings \(d_1\) and \(d_2\) are epimorphic for all \(t \in \mathbb{Z}\) and from (3.3) we get the exact sequence
\[
0 \longrightarrow H^q(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(\log D)(t)) \longrightarrow H^q(D, \Omega_{D}^{-1}(t)) \xrightarrow{d_2} H^{q+1}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(t)) \longrightarrow 0.
\]
Under these assumptions holds \(H^q(D, \Omega_{D}^{-1}(t)) = 0\) if \(q \neq r - 1\) or \(t \neq 0\) (cf. [1]). In case \(q = r - 1, t = 0\) we know that \(d_2\) is an isomorphism by Lemma 3.1 since \(2(r - 1) < N - 1\). Therefore, one has
\[
H^q(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(\log D)(t)) = 0 \text{ for } 0 < q , 0 < r \text{ and } q + r < N.
\]
For \(q < N, r < N, q + r > N\) we use the Serre duality to show statement (a). The case \(r = N\) is trivial since \(\Omega_{\mathbb{P}^N}(\log D) \cong \mathcal{O}_{\mathbb{P}^N}(m - N - 1)\).
If \(r \geq 2\) and \(q + r = N\) then the mappings \(d_1\) and \(d_2\) are epimorphic, i.e.
\[
\dim H^{N-1}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(\log D)(t)) = \dim H^{N-1}(D, \Omega_{D}(t)) - \dim H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(\log D)(t)).
\]
In the case \(r = 1\) and \(q = N - 1\) one has
\[
\dim H^{N-1}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(\log D)(t)) = \dim H^{N-1}(D, \mathcal{O}_D(t)) - \dim H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(\log D)(t)) + H^N(\mathbb{P}^N, \Omega_{\mathbb{P}^N}(\log D)(t)).
\]
Applying Theorem 3.2, Lemma 2.1 and the results in [1] delivers (b) and (c). \(\square\)

3.2. \(T\)-symmetric Tensor Differential Forms. Let \(T\) be a Young tableau with \(r\) boxes. We study the sheaf \(\Omega^T(\log D) = (\Omega^1(\log D))^T\) on \(\mathbb{P}^N\) and begin with a free resolution of the sheaf \(\Omega^1(\log D)\).

Lemma 3.4. Let \(D \subset \mathbb{P}^N\) be a nonsingular, irreducible, algebraic hypersurface of degree \(m \geq 2\) defined by the equation \(F = 0\).
Then there exists a short exact sequence
\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-m) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1) \longrightarrow \mathcal{O}_D(\log D) \longrightarrow 0. \quad (3.4)
\]
If \(D\) is a hyperplane, i.e. \(m = 1\), we have \(\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{P}^N}(-1)\).

Proof. Let \(U_i = \{x_i \neq 0\} \subset \mathbb{P}^N\) and let \(U \subseteq \mathbb{P}^N\) be an arbitrary open affine subset. We are going to show that there is an exact sequence
\[
0 \longrightarrow \Gamma(U, \mathcal{O}_{\mathbb{P}^N}(-m)) \longrightarrow \Gamma(U, \mathcal{O}_{\mathbb{P}^N}(-1)) \longrightarrow \Gamma(U, \mathcal{O}_D(\log D)) \longrightarrow 0.
\]
For sections \(f_0, \ldots, f_N \in \Gamma(U, \mathcal{O}(-1))\) we put \(g := -\frac{1}{m} \sum_{\mu=0}^{N} x_{\mu} f_{\mu} \in \Gamma(U, \mathcal{O})\).
Let \(F_j = \frac{\partial F}{\partial x_j}\) denotes the partial derivatives of \(F\). The mapping \(\beta\) is defined by
\((f_0, \ldots, f_N) \mapsto \omega\), where the differential form \(\omega\) on \(U \cap U_i\) is given by
\[
\omega = \omega_i := \sum_{\nu \neq i}^N \left( f_\nu + g \cdot \frac{F_\nu}{F} \right) \cdot x_i \cdot \frac{d x_\nu}{x_i}.
\]

One easily verifies that \(\omega\) is a section of \(\Omega_{\mathbb{P}^N}^1 (\log D)\) on \(U\) and it holds, in particular, \(\omega_i = \omega_j\) for any \(i, j \in \{0, 1, \ldots, N\}\). For a section \(\delta \in \Gamma(U, \mathcal{O}(-m))\) let \(f_\nu = \delta \cdot F_\nu\) for all \(\nu = 0, 1, \ldots, N\) which implies that \(f_\nu \in \Gamma(U, \mathcal{O}(-1))\) and \(g = -\delta \cdot F\). Finally, we have \(\ker \beta = \{(\delta \cdot F_0, \ldots, \delta \cdot F_N)\} \cong \Gamma(U, \mathcal{O}_{\mathbb{P}^N}(-m))\), which yields the claim for \(m \geq 2\).

In the last part we have to show the statement of Lemma 3.4 in case \(m = 1\). Let \(D \subset \mathbb{P}^N\) be the hyperplane satisfying the equation \(x_N = 0\), and let \(U \subset \mathbb{P}^N\) be an open subset. For given sections \(f_0, \ldots, f_{N-1} \in \Gamma(U, \mathcal{O}(-1))\) let \(\omega\) be the differential form, which has on \(U \cap U_i\), \(i = 0, \ldots, N-1\), the representation
\[
\omega = \omega_i = \sum_{\nu \neq i}^{N-1} f_\nu \cdot x_i \cdot \frac{d x_\nu}{x_i} - \left( \sum_{\mu = 0}^{N-1} f_\mu \cdot x_\mu \right) \cdot \frac{x_i}{x_N} \cdot \frac{d x_N}{x_i},
\]
respectively on \(U \cap U_N\),
\[
\omega = \omega_N = \sum_{\nu = 0}^{N-1} f_\nu \cdot x_N \cdot \frac{d x_\nu}{x_N}.
\]

Then \(\omega\) is a section of \(\Omega_{\mathbb{P}^N}^1 (\log D)\) on \(U\), and the mapping \((f_0, \ldots, f_{N-1}) \mapsto \omega\) becomes an isomorphism of \(\Gamma(U, \mathcal{O}_{\mathbb{P}^N}^N(-1))\) onto \(\Gamma(U, \Omega_{\mathbb{P}^N}^1 (\log D))\).

**Lemma 3.5.** Let \(T\) be a Young tableau with \(r\) boxes and the row lengths \(l_1, l_2, \ldots, l_d\), set \(t_i := r + l_i - i\) for all \(i \geq 1\) (\(l_i = 0\) if \(i > d\)) and assume \(d = \text{depth} T \leq N\).

Then the following sequence is exact for \(m \geq 2\):
\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(d \cdot (1 - m) - r) \overset{a_d}{\longrightarrow} \mathcal{O}_{\mathbb{P}^N}((d - 1) \cdot (1 - m) - r) \overset{a_{d-1}}{\longrightarrow} \mathcal{O}_{\mathbb{P}^N}((d - 2) \cdot (1 - m) - r) \overset{a_{d-2}}{\longrightarrow} \mathcal{O}_{\mathbb{P}^N}(1 - m - r) \overset{a_1}{\longrightarrow} 0
\]
with the integers
\[
b_s = \left( \prod_{i=1}^N i! \right)^{-1} \cdot \sum_{1 \leq t_1 < \ldots < t_s \leq d} \Delta(t_1, t_2, \ldots, t_s, t_s - 1, \ldots, t_N, t_N + 1).
\]

where \(\Delta\) denotes the Vandermonde determinant.

In the case \(s = 0\) we have
\[
b_0 = \left( \prod_{i=1}^N i! \right)^{-1} \cdot \prod_{1 \leq i < j \leq N+1} (l_i - l_j + j - i) = \left( \prod_{i=1}^N i! \right)^{-1} \cdot \Delta(t_1, t_2, \ldots, t_N, t_N + 1).
\]

For \(m = 1\) (\(D\) is a hyperplane) it holds
\[
\Omega_{\mathbb{P}^N}^T (\log D) \cong \bigoplus_{r \leq \text{rk}(\Omega_{\mathbb{P}^N}^1)} \mathcal{O}_{\mathbb{P}^N}(-r)
\]
with

\[ \text{rk}(\Omega_{P,N}^T) = \prod_{1 \leq i < j \leq N} \left( \frac{l_i - l_j}{j - i} + 1 \right) = \sum_{i=0}^{d} (-1)^i \cdot b_i \]

\[ = \left( \prod_{i=1}^{N-1} i! \right)^{-1} \cdot \Delta(t_1, t_2, \ldots, t_N). \]

**Proof.** The T-Power of (3.4) yields the claim for \( m \geq 2 \) (cf. [3]) and Lemma 3.4 shows the case \( m = 1 \). \( \square \)

**Theorem 3.6.** Let \( T \) be a Young tableau with \( r \) boxes and with \( d = \text{depth} T \) rows.

(a) \( \chi(\mathbb{P}^N, \Omega_{P,N}^T (\log D)(t)) = \frac{1}{N!} \cdot \sum_{i=0}^{d} (-1)^i \cdot b_i \cdot \prod_{j=1}^{N} (t - i \cdot (m - 1) + j - r) \)

(b) For depth \( T < N \) one has:

\[ \dim H^0(\mathbb{P}^N, \Omega_{P,N}^T (\log D)(t)) = \sum_{i=0}^{d} (-1)^i \cdot b_i \cdot \left( t - i \cdot (m - 1) + N - r \right) \]

and therefore: \( H^0(\mathbb{P}^N, \Omega_{P,N}^T (\log D)(t)) \neq 0 \iff t \geq r \)

(c) Let \( d = \text{depth} T = N \) and let \( l_N \) be the number of columns of \( T \) with the length \( N \). We denote by \( T' \) the Young tableau which is given by \( T \) without these columns of length \( N \). Then depth \( T' < N \) and it holds \( \Omega_{P,N}^T (\log D)(t) \cong \Omega_{P,N}^T (\log D)(t + l_N \cdot (m - N - 1)) \).

If \( T \) is a rectangle with \( N \) rows and \( l \) columns, then we have \( \Omega_{P,N}^T (\log D)(t) \cong O_{P,N}(t + l \cdot (m - N - 1)) \).

(d) For \( 1 \leq q < N - d \) we get \( H^q(\mathbb{P}^N, \Omega_{P,N}^T (\log D)(t)) = 0 \) for all \( t \in \mathbb{Z} \).

(e) Let \( d_l \) be the length of the last column of \( T \). Then it holds:

\[ H^q(\mathbb{P}^N, \Omega_{P,N}^T (\log D)(t)) = 0 \text{ for } N - d_l < q < N \text{ and } \forall t \in \mathbb{Z}. \]

**Proof.** The short exact sequences of (3.5) yields

\[ 0 \rightarrow \bigoplus_{b_d} O_{P,N} (d \cdot (1 - m) - r) \rightarrow \bigoplus_{b_{d-1}} O_{P,N} ((d - 1) \cdot (1 - m) - r) \rightarrow \text{Im} \alpha_{d-1} \rightarrow 0 \]

\[ 0 \rightarrow \text{Im} \alpha_{d-1} \rightarrow \bigoplus_{b_{d-2}} O_{P,N} ((d - 2) \cdot (1 - m) - r) \rightarrow \text{Im} \alpha_{d-2} \rightarrow 0 \]

\[ \vdots \]

\[ 0 \rightarrow \text{Im} \alpha_2 \rightarrow \bigoplus_{b_2} O_{P,N} (1 - m - r) \rightarrow \text{Im} \alpha_1 \rightarrow 0 \]

\[ 0 \rightarrow \text{Im} \alpha_1 \rightarrow \bigoplus_{b_0} O_{P,N} (-r) \rightarrow \Omega_{P,N}^T (\log D) \rightarrow 0, \]

where \( H^q(\mathbb{P}^N, O_{P,N}(t)) = 0 \) for \( 1 \leq q \leq N - 1 \) and for all \( t \in \mathbb{Z} \). This implies
$H^q(\mathbb{P}^N, \text{Im} \alpha_1(t)) = 0$ for $1 \leq q \leq N - 1 + i - d$ and hence, we have in case $d < N$
\[ \dim H^0(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}(\log D)(t)) = b_0 \cdot \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t - r)) - \dim H^0(\mathbb{P}^N, \text{Im} \alpha_1(t)) \]
\[ \dim H^0(\mathbb{P}^N, \text{Im} \alpha_1(t)) = b_1 \cdot \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t + 1 - m - r)) - \dim H^0(\mathbb{P}^N, \text{Im} \alpha_2(t)) \]
\[ \vdots \]
\[ \dim H^0(\mathbb{P}^N, \text{Im} \alpha_{d-1}(t)) = b_{d-1} \cdot \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t + (d - 1) \cdot (1 - m) - r)) - b_d \cdot \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t + d \cdot (1 - m) - r)). \]

This shows (b). For $d = N$ we already know that
\[ \Omega^N_{\mathbb{P}^N}(\log)(t) \cong \Omega^N_{\mathbb{P}^N}(\log D) \otimes \cdots \otimes \Omega^N_{\mathbb{P}^N}(\log D) \otimes \Omega^N_{\mathbb{P}^N}(\log D)(t) \]
\[ \cong \Omega^N_{\mathbb{P}^N}(\log D)(t + l_N \cdot (m - N - 1)), \]

which proves assertion (c). In order to prove (d), we consider again the short exact sequences of (3.5) and obtain
\[ H^q(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}(\log D)(t)) = 0 \]
\[ H^{q+1}(\mathbb{P}^N, \text{Im} \alpha_1(t)) = 0 \]
\[ H^{q+2}(\mathbb{P}^N, \text{Im} \alpha_2(t)) = 0 \]
\[ \vdots \]
\[ H^{q+d-1}(\mathbb{P}^N, \text{Im} \alpha_{d-1}(t)) = 0 \]

This implies $H^q(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}(\log D)(t)) = 0$ for $1 \leq q \leq N - d - 1$ and $\forall t \in \mathbb{Z}$.

The last statement can be proven by Serre duality which means
\[ \dim H^q(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}(\log D)(t)) = \dim H^{N-q}(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}(\log D)(-t - m - (l - 1) \cdot (m - N - 1))), \]

where $\text{depth } T^* = N - d_1 < N$. Note if we use (b) with $T^*$ instead of $T$, we obtain a formula for $\dim H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}(\log D)(t))$.

### 3.3. Symmetric Differential Forms

Let $T$ be a Young tableau with $r$ boxes and only one row, i.e. depth $T = 1$. We will specify the dimensions of $H^q(\mathbb{P}^N, S^r \Omega^1(\log D)(t))$ and consider the following exact sequence (cf. Lemma 3.5)
\[ 0 \rightarrow \bigoplus_{b_1} \mathcal{O}_{\mathbb{P}^N}(-m + 1 - r) \rightarrow \bigoplus_{b_0} \mathcal{O}_{\mathbb{P}^N}(-r) \rightarrow S^r \Omega^1_{\mathbb{P}^N}(\log D) \rightarrow 0 \quad (3.6) \]

with the integers $b_0 = \binom{N + r}{N}$ and $b_1 = \binom{N + r - 1}{N}$.

**Theorem 3.7.** Let $N \geq 2$. Then one has:

(a) $\chi(\mathbb{P}^N, S^r \Omega^1_{\mathbb{P}^N}(\log D)(t))$
\[ = \frac{1}{N!} \cdot \binom{N + r}{N} \cdot \prod_{j=1}^{N} (t - r + j) - \frac{1}{N!} \cdot \binom{N + r - 1}{N} \cdot \prod_{i=1}^{N} (t - m + 1 - r + i). \]
implies (b) and (c). Using the Serre Duality yields

\[ \dim H^0(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) \]

\[ = \binom{N + r}{N} \cdot \binom{t - r + N}{N} - \binom{N + r - 1}{N} \cdot \binom{t - m + 1 - r + N}{N}. \]

(c) For \( 1 \leq q \leq N - 2 \) it holds: \( H^q(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) = 0 \) for all \( t \in \mathbb{Z} \).

(d) \( \dim H^{N-1}(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) \)

\[ = \sum_{i=0}^{N-1} (-1)^i \cdot \bar{b}_i \cdot \binom{-t + r + 2 + i \cdot (m + 1) - 1}{N} \cdot \binom{N + r + 1 - i}{N} \cdot \binom{t - m + r + 2 - 1}{N}. \]

with the integers

\[ \bar{b}_i = \frac{N + r}{N + r} \cdot \binom{N + r}{N - 1 - i} \cdot \binom{N + r}{N - r - 1 - i} \cdot \binom{N + r + 1 - i}{N}. \] \hspace{1cm} (3.7)

Proof. (a) follows directly from (3.6) and the additivity of the Euler characteristic. We consider (3.6) together with the corresponding cohomology sequence and know that \( H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = 0 \) for any \( q \in \{1, \ldots, N - 1\} \) and for all \( t \in \mathbb{Z} \), which implies (b) and (c). Using the Serre Duality yields

\[ \dim H^N(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) \]

\[ = \dim H^0(\mathbb{P}^N, \Omega^{T^*}(\log D)(-t + (r + 1) \cdot (N + 1) - r \cdot m)) \]

where \( T^* \) is a rectangle with depth \( T^* = N - 1 \) rows and length \( T^* = r \) columns and with the associated integers \( \bar{b}_i \) in (3.7) (cf. Lemma 3.5). Theorem 3.6(b) delivers the formula for \( \dim H^0(\mathbb{P}^N, \Omega^{T^*}(\log D)(-t + (r + 1) \cdot (N + 1) - r \cdot m)) \).

Finally, one gets easily the dimension \( \dim H^{N-1}(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) \) from the long cohomology sequence. \( \square \)

Corollary 3.8. For \( N \geq 2 \) we obtain:

(a) \( H^0(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) \neq 0 \iff t \geq r. \)

(b) \( \dim H^0(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) = \chi(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) \)

if \( t \geq m + r - N - 1. \)

(c) \( H^N(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) = 0 \iff t \geq -r(m - 2) - N. \)

(d) \( H^{N-1}(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) = 0 \) if \( t \geq m + r - N - 1. \)

Proof. Obviously, the proof follows from Theorem 3.7. \( \square \)

Theorem 3.9. Let \( N \geq 2 \) and let \( D \) be a hyperplane, that is, \( m = 1. \) Then one has

(a) \( \dim H^0(\mathbb{P}^N, S^r \Omega_{\mathbb{P}^N}^1 (\log D)(t)) = \binom{N + r - 1}{N - 1} \cdot \binom{t - r + N}{N}. \)
(b) For $1 \leq q \leq N - 1$ it holds: $H^q(\mathbb{P}^N, S^r \Omega^1_{\mathbb{P}^N}(\log D)(t)) = 0 \forall t \in \mathbb{Z}.$

(c) $\dim H^N(\mathbb{P}^N, S^r \Omega^1_{\mathbb{P}^N}(\log D)(t)) = \binom{N + r - 1}{N - 1} \cdot \binom{-t + r - 1}{N}.$

Proof. $S^r \Omega^1_{\mathbb{P}^N}(\log D) \cong \bigoplus_{N-1}^N \mathcal{O}_{\mathbb{P}^N}(-r).$ \qed

4. Complete Intersections $Y \subset \mathbb{P}^N$

Let $Y = H_1 \cap \ldots \cap H_{N-n} \subset \mathbb{P}^N$ be a nonsingular, irreducible, complete intersection of algebraic hypersurfaces $H_i \subset \mathbb{P}^N,$ where $H_i$ is given by the equation $F_i = 0$ with $\deg F_i = m_i.$ We denote by $n$ the dimension of $Y.$ Let $D$ be a prime divisor on $Y,$ which is defined by the equation $D = Y \cap H$ with a hypersurface $H : F = 0.$ The degree of $H$ is $m.$ In the following, we abbreviating denote $c = N - n = \text{codim} Y$ and assume $n \geq 2.$ Let $X$ be a further complete intersection which is described by $X = H_1 \cap \ldots \cap H_{c-1}.$ Here $\dim X = n + 1$ and $Y = X \cap H_c.$ There exists also a divisor $D^* = X \cap H$ on $X.$ Assume that the hypersurfaces $H_1, \ldots, H_{N-n}$ and $H$ lie in general position, i.e. for instance $X = H_1 \cap \ldots \cap H_{c-1} \subset \mathbb{P}^N$ and the prime divisors $D$ on $Y$ and $D^*$ on $X$ are nonsingular, irreducible, complete intersections, too.

4.1. Alternating Differential Forms. In case $r = n$ we obtain $\Omega^r_Y = \omega_Y \cong \mathcal{O}_Y(\sum_{i=1}^c m_i - N - 1)$ which implies

$\Omega^r_Y(\log D) \cong \Omega^r_Y(m) \cong \mathcal{O}_Y(\sum_{i=1}^c m_i - N - 1 + m),$ 

where $D = Y \cap H$ with $\deg H = m.$ The dimensions of $H^q(Y, \Omega^r_Y(\log D)(t)) = H^q(Y, \mathcal{O}_Y(\sum_{i=1}^c m_i - N - 1 + m + t))$ are well known:

If $1 \leq q \leq n - 1$ then $H^q(Y, \mathcal{O}_Y(t)) = 0 \forall t \in \mathbb{Z}.$

\[
\dim H^0(Y, \mathcal{O}_Y(t)) = \binom{t + N}{N} + \\
+ \sum_{j=1}^c (-1)^j \cdot \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq c} \binom{t + N - m_{i_1} - m_{i_2} - \ldots - m_{i_j}}{N} \\
\dim H^n(Y, \mathcal{O}_Y(t)) = \dim H^0(Y, \mathcal{O}_Y(-t + m_1 + m_2 + \ldots + m_c - N - 1)),
\]

(cf. e.g. [1] or the proof of Lemma (4.4) in the present paper).

We study the cohomology groups $H^q(Y, \Omega^r_Y(\log D)(t))$ with $r < \dim Y = n:

Lemma 4.1. The following sequences are exact.

(a) $0 \to \mathcal{O}_X(-m_c) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ \hspace{1cm} (4.1)

(b) $0 \to \Omega^r_X(\log D^*)(-m_c) \xrightarrow{\alpha} \Omega^r_X(\log D^*) \xrightarrow{\beta} \mathcal{O}_Y \otimes \mathcal{O}_X \Omega^r_X(\log D^*) \to 0$ \hspace{1cm} (4.2)

(c) $0 \to \Omega_Y^{r-1}(\log D)(-m_c) \xrightarrow{\gamma} \mathcal{O}_Y \otimes \mathcal{O}_X \Omega^r_X(\log D^*) \xrightarrow{\delta} \Omega^r_Y(\log D) \to 0$ \hspace{1cm} (4.3)
Proof. Notice, for \( r = 1 \) we have to substitute \( \Omega_Y^{-1}(\log D) \) by the structure sheaf \( \mathcal{O}_Y \). The composition \( \delta \circ \beta \) is the restriction of the differential forms on \( X \) to the subvariety \( Y \subset X \). Obviously, the sequence (4.1) is exact and (4.2) results by multiplication of (4.1) with the locally free sheaf \( \Omega_X^r(\log D^*) \). We will show that (4.3) is also an exact sequence. Let \( U \subseteq X \) be an open subset of \( X \) and let \( V = Y \cap U \) be an open, nonempty subset of \( Y \). Without loss of generality we assume \( U \subseteq \{ x_i \neq 0 \} \). Moreover, we suppose the existence of local parameters \( u_1, \ldots, u_{n-1}, u_n = F_{x_i}, u_{n+1} = \frac{F}{x_i} \) of \( X \) on \( U \) such that their restriction to \( Y \) are also local parameters \( v_1 = \varphi^*(u_1), \ldots, v_{n-1} = \varphi^*(u_{n-1}), v_n = \varphi^*(u_n) = \frac{F}{x_i} \) of \( Y \) on \( V \). Then \( \Gamma(V, \mathcal{O}_Y \otimes \Omega_X^r(\log D^*)) \) is a free \( \Gamma(V, \mathcal{O}_Y) \)-module whose rank is equal to \( \binom{n+1}{r} \). Let \( \omega \in \Gamma(V, \mathcal{O}_Y \otimes \Omega_X^r(\log D^*)) \) be a section of the form

\[
\omega = \sum_{i_r=1}^{n-1} f_{i_1, \ldots, i_r} \frac{d u_i}{u_i} \wedge \ldots \wedge d u_{i_r},
\]

where \( f_{i_1, \ldots, i_r} \in \Gamma(V, \mathcal{O}_Y) \). The homomorphism \( \delta \) is defined as follows:

\[
\delta(\omega) = \sum_{i_r=1}^{n-1} f_{i_1, \ldots, i_r} \frac{d v_i}{v_i} \wedge \ldots \wedge d v_{i_r} + \sum_{i_r=1}^{n-1} f_{i_1, \ldots, i_r, n} \frac{d v_i}{v_i} \wedge \ldots \wedge d v_{i_r} \wedge d u_n \wedge \frac{d u_n}{u_n},
\]

which means that \( \delta(\omega) \in \Gamma(V, \Omega_Y^r(\log D)) \). The kernel of \( \delta \) is given by

\[
\ker \delta = \left\{ \sum_{i_r=1}^{n-1} f_{i_1, \ldots, i_r, n+1} \frac{d u_i}{u_i} \wedge \ldots \wedge d u_{i_r} \wedge d u_n + \sum_{i_r=1}^{n-1} f_{i_1, \ldots, i_r, n+1} \frac{d u_i}{u_i} \wedge \ldots \wedge d u_{i_r} \wedge \frac{d u_n}{u_n} \wedge d u_{n+1} \right\},
\]

where \( \ker \delta \subseteq \Gamma(V, \mathcal{O}_Y \otimes \Omega_X^r(\log D^*)) \). In order to show that the kernel of \( \delta \) is isomorphic to \( \Gamma(V, \Omega_Y^{-1}(\log D)(-m_c)) \), we consider the following homomorphisms

\[
\ker \delta \xrightarrow{\alpha} \Gamma(V, \mathcal{O}_Y(-m_c) \otimes \Omega_X^{-1}(\log D^*)) \xrightarrow{\beta} \Gamma(V, \Omega_Y^{-1}(\log D)(-m_c)).
\]

Let \( \xi \in \ker \delta \) be any element. The mappings \( \alpha \) and \( \beta \) are illustrated by

\[
\alpha(\xi) = \frac{1}{x_i^{m_c}} \sum_{i_r=1}^{n-1} f_{i_1, \ldots, i_r, n+1} \frac{d u_i}{u_i} \wedge \ldots \wedge d u_{i_r},
\]

\[
\beta(\xi) = \frac{1}{x_i^{m_c}} \sum_{i_r=1}^{n-1} f_{i_1, \ldots, i_r, n+1} \frac{d u_i}{u_i} \wedge \ldots \wedge d u_{i_r} \wedge \frac{d u_n}{u_n}.
\]
Since \( x_i^{m_c} \cdot d u_{n+1} = x_i^{m_c} \cdot d \frac{F_{x_i}}{y} \) is a global section of the sheaf \( O_Y(m_c) \otimes O_X \Omega_X^1 \) respectively,

\[
\tilde{\beta}(\tilde{\alpha}(\xi)) = \frac{1}{x_i^{m_c}} \sum_{i_r=1}^{n-1} f_{i_1,\ldots,i_r-1,n+1} \ d v_{i_1} \wedge \ldots \wedge d v_{i_r-1} \\
+ \frac{1}{x_i^{m_c}} \sum_{i_r=1}^{n-1} f_{i_1,\ldots,i_r-2,n,n+1} \ d v_{i_1} \wedge \ldots \wedge d v_{i_r-2} \wedge \frac{d v_n}{v_n}.
\]

Since \( x_i^{m_c} \cdot d u_{n+1} = x_i^{m_c} \cdot d \frac{F_{x_i}}{y} \) is a global section of the sheaf \( O_Y(m_c) \otimes O_X \Omega_X^1 \) respectively, \( \tilde{\alpha} \) and \( \tilde{\beta} \) are independent of the index \( i \) with \( U \subseteq U_i \) and independent of the choice of the local parameters \( u_1, \ldots, u_{n-1} \). One can easily see that \( \tilde{\alpha} \) and \( \tilde{\beta} \) are monomorphic. The mapping \( \tilde{\beta} \) is the restriction from \( X \) to \( Y \) which obviously is epimorphic. While \( \tilde{\alpha} \) is generally not epimorphic, any element of \( \Gamma(V, \Omega_Y^{-1}(\log D)(-m_c)) \) has a preimage in \( \ker \delta \). We can represent an element of \( \Gamma(V, \Omega_Y^{-1}(\log D)(-m_c)) \) by the form \( \tilde{\beta}(\tilde{\alpha}(\xi)) \) with functions \( f_{i_1,\ldots,i_r} \in \Gamma(V,O_Y) \).

In order to find a preimage in \( \ker \delta \), we take the local parameters \( u_i \) on \( X \) and multiply with \( x_i^{m_c} \cdot d u_{n+1} \). This proves that the composition \( \tilde{\beta} \circ \tilde{\alpha} \) is isomorphic, the sequence (4.3) is exact. \( \square \)

By means of these exact sequences we are going to prove recursion formulas about the dimensions of the cohomology groups \( H^q(Y, \Omega_Y^r(\log D)(t)) \). As mentioned above, for \( r = n \) these dimensions are known.

**Theorem 4.2.**

(a) \( \chi(Y, \Omega_Y^r(\log D)(t)) = \chi(X, \Omega_X^r(\log D^*)(t)) \)

\[ -\chi(X, \Omega_X^r(\log D^*)(t-m_c)) - \chi(Y, \Omega_Y^{r-1}(\log D)(t-m_c)) \] for \( r \geq 1 \)

In the case \( r = 1 \) one has to substitute \( \Omega_Y^{r-1}(\log D) \) by the structure sheaf \( O_Y \).

(b) Let \( 0 < q < n \), \( q + r \neq n \) and \( r \geq 0 \).

Then one has \( H^q(Y, \Omega_Y^r(\log D)(t)) = 0 \) for any \( t \in \mathbb{Z} \).

(c) \( \dim H^0(Y, \Omega_Y^r(\log D)(t)) \)

\[ = \dim H^0(X, \Omega_X^r(\log D^*)(t)) - \dim H^0(X, \Omega_X^{r-1}(\log D^*)(t-m_c)) \]

\[ - \dim H^0(Y, \Omega_Y^{r-1}(\log D)(t-m_c)) \] for \( 0 < r < n \)

(d) \( \dim H^n(Y, \Omega_Y^r(\log D)(t)) = \dim H^n(X, \Omega_X^{n-r}(\log D^*)(-t-m)) \)

\[ - \dim H^0(X, \Omega_X^{n-r}(\log D^*)(-t-m_c-m)) \]

\[ - \dim H^0(Y, \Omega_Y^{n-r-1}(\log D)(-t-m_c-m)) \]

(e) \( \dim H^1(Y, \Omega_Y^{n-1}(\log D)(t)) \)

\[ = \dim H^0(Y, \Omega_Y^{n-1}(\log D)(t)) + \dim H^0(Y, \Omega_Y^r(\log D)(t+m_c)) \]

\[ + \dim H^0(X, \Omega_X^r(\log D^*)(t)) - \dim H^0(X, \Omega_X^{r-1}(\log D^*)(t+m_c)) \]

\[ - \dim H^1(X, \Omega_X^r(\log D^*)(t)) + \dim H^1(X, \Omega_X^{r-1}(\log D^*)(t+m_c)) \]
(f) $\dim H^{n-r}(Y, \Omega^r_Y (\log D)(t))$
\begin{align*}
= \dim H^{n-r-1}(Y, \Omega^{r+1}_Y (\log D)(t + m_c)) &- \dim H^{n-r}(X, \Omega^{r+1}_X (\log D^*)(t)) \\
+ \dim H^{n-r}(X, \Omega^{r+1}_X (\log D^*)(t + m_c)) & \text{ for } 2 \leq r < n
\end{align*}

**Proof.** Under the additional condition $q + r < n$ the proof of (b) will be shown by complete induction with respect to $c = \text{codim} Y$ and $r$. Then the case $q + r > n$ follows directly from the Serre duality. If $c = 0$, i.e. $Y = \mathbb{P}^N$, Theorem 3.3 implies $H^q(Y, \Omega^r_Y (\log D)(t)) = 0$ for $0 < q < N$ and $q + r \neq N$.

If $r = 0$ then we get $H^q(Y, \mathcal{O}_Y (t)) = 0$ for $0 < q < n$ (cf. e.g. [2, Lemma 1]).

In particular, we have the following induction assumption ($c-1 = \text{codim} X$):

(i) From $q, r \in \mathbb{N}$, $0 < q$, $0 < r$ and $q + r < n + 1$ it follows

$$H^q(X, \Omega^r_X (\log D^*)(t)) = 0 \text{ for all } t \in \mathbb{Z}.$$ 

Now assume $0 < q$, $0 \leq r$ and $q + r < n$. From (4.2) we get the exact sequence

$$\cdots \longrightarrow H^q(X, \Omega^r_X (\log D^*)(t)) \longrightarrow H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X (\log D^*)) \longrightarrow$$
\begin{align*}
\longrightarrow H^{q+1}(X, \Omega^r_X (\log D^*)(t - m_c)) & \longrightarrow \cdots.
\end{align*}

Since $0 < q$, $q + 1 + r < n + 1$ we have by induction assumption (i):

$$H^q(X, \Omega^r_X (\log D^*)(t)) = 0 \text{ and } H^{q+1}(X, \Omega^r_X (\log D^*)(t - m_c)) = 0.$$ 

Hence,

$$H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X (\log D^*)) = 0 \text{ for } 0 < q, \ q + r < n \text{ and any } t \in \mathbb{Z}.$$ 

Now, let $r > 0$ be a fixed integer. We use the following induction assumption:

(ii) If $0 < q$ and $q + r - 1 < n$ then $H^q(Y, \Omega^{r-1}_Y (\log D)(t)) = 0$ for all $t \in \mathbb{Z}$.

To prove: If $0 < q$ and $q + r < n$ then $H^q(Y, \Omega^r_Y (\log D)(t)) = 0$ for all $t \in \mathbb{Z}$.

Let $0 < q$, $q + r < n$. We consider the exact sequence which is given by (4.3)

$$\cdots \longrightarrow H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X (\log D^*)) \longrightarrow H^q(Y, \Omega^r_Y (\log D)(t)) \longrightarrow$$
\begin{align*}
\longrightarrow H^{q+1}(Y, \Omega^{r-1}_Y (\log D)(t - m_c)) & \longrightarrow \cdots.
\end{align*}

By (ii) one has $H^{q+1}(Y, \Omega^{r-1}_Y (\log D)(t - m_c)) = 0$ for all $t \in \mathbb{Z}$ since $q + 1 + r - 1 = q + r < n$ (and $q + 1 < n$). Furthermore, we know that

$$H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X (\log D^*)) = 0 \text{ for any } t \in \mathbb{Z} \text{ because of } 0 < q, \ q + r < n.$$ 

This implies $H^q(Y, \Omega^r_Y (\log D)(t)) = 0$ for $0 < q < n$ and $q + r < n$ for any $t \in \mathbb{Z}$.

For the proof of (c) we first consider the exact sequence from (4.2)

$$0 \longrightarrow H^0(X, \Omega^r_X (\log D^*)(t - m_c)) \longrightarrow H^0(X, \Omega^r_X (\log D^*)(t)) \longrightarrow$$
\begin{align*}
\longrightarrow H^0(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X (\log D^*)) & \longrightarrow \\
\longrightarrow H^1(X, \Omega^r_X (\log D^*)(t - m_c)) & \longrightarrow \cdots,
\end{align*}

and apply (i) which yields $H^1(X, \Omega^r_X (\log D^*)(t - m_c)) = 0$ as $1 + r < n + 1 = \dim X$.

Because of (4.3) one gets the exact sequence

$$0 \longrightarrow H^0(Y, \Omega^{r-1}_Y (\log D)(t - m_c)) \longrightarrow H^0(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X (\log D^*)) \longrightarrow$$
\begin{align*}
H^0(Y, \Omega^r_Y (\log D)(t)) & \longrightarrow H^1(Y, \Omega^{r-1}_Y (\log D)(t - m_c)) \longrightarrow \cdots,
\end{align*}

and due to $1 + r - 1 = r < n$ one has $H^1(Y, \Omega^{r-1}_Y (\log D)(t - m_c)) = 0$. Statement (c) can be read from (4.4) and (4.5). Assertion (d) can easily be shown by Serre duality.
We need to show that (4.7) is an exact sequence. Let (4.6) be the T-Power of the following short exact sequence (cf. [3]):

Lemma 4.3. There exists following exact sequence:

For simplification we set furthermore: 

for all \( A \in M(T) \) by the structure sheaf \( \mathcal{O}_Y \).

Lemma 4.3. There exists following exact sequence:

\[
0 \to E_T^\mu \xrightarrow{\beta_\mu} E_T^{\mu-1} \xrightarrow{\beta_{\mu-1}} \ldots \xrightarrow{\beta_2} E_T^1 \xrightarrow{\beta_1} \Omega_{\mathbb{P}^N|Y}^T(\log D^*)(t(A)) \to 0.
\]  

(4.6) 

Proof. (4.6) is the T-Power of the following short exact sequence (cf. [3]):

\[
0 \to \bigoplus_{i=1}^c \mathcal{O}_Y(-m_i) \xrightarrow{\alpha} \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega_{\mathbb{P}^N}^1(\log D^*) \xrightarrow{\beta} \Omega_{\mathbb{P}^N}^1(\log D) \to 0. 
\]  

(4.7) 

We need to show that (4.7) is an exact sequence. Let \( U \subseteq \mathbb{P}^N \) be an open subset. Without loss of generality we put \( U = \bigcup_i U_i = \{ x_i \neq 0 \} \). Assume that there exist local parameters \( \frac{F_1}{x_i}, \ldots, \frac{F_c}{x_i}, u_1, \ldots, u_{n-1}, \frac{F}{x^{n-1}_i} \) of \( \mathbb{P}^N \) on \( U \) such that the restrictions 

\[
v_1 = \varphi^*(u_1), \ldots, v_{n-1} = \varphi^*(u_{n-1}), v_n = \varphi^*(\frac{F}{x^{n-1}_i})
\]  

are local parameters of \( Y \) on
The homomorphism $\beta$ defined by the span $U$ which is isomorphic and independent of the index $i$.

For an arbitrary Young tableau $T$, there exists the following exact sequence which is called the Koszul complex:

$$0 \longrightarrow \Omega_{\mathcal{P}_N}^r(\log D^*) \longrightarrow \bigoplus_{1 \leq i \leq c} \Omega_{\mathcal{P}_N}^r(\log D^*)(-\sum_{j=1}^c m_j + m_i) \stackrel{\alpha_i}{\longrightarrow} \bigoplus_{1 \leq i \leq c} \Omega_{\mathcal{P}_N}^r(\log D^*)(-m_i) \longrightarrow 0.$$ (4.8)

**Lemma 4.4.** For an arbitrary Young tableau $T'$ there exists the following exact sequence

$$0 \longrightarrow \Omega_{\mathcal{P}_N}^r(\log D^*) \longrightarrow \bigoplus_{1 \leq i \leq c} \Omega_{\mathcal{P}_N}^r(\log D^*)(-\sum_{j=1}^c m_j + m_i) \stackrel{\alpha_i}{\longrightarrow} \bigoplus_{1 \leq i \leq c} \Omega_{\mathcal{P}_N}^r(\log D^*)(-m_i) \longrightarrow 0.$$ (4.8)

**Proof.** We consider the following exact sequence which is called the Koszul complex:

$$0 \longrightarrow \mathcal{O}_{\mathcal{P}_N}(-\sum_{i=1}^c m_i) \stackrel{\alpha_i}{\longrightarrow} \bigoplus_{1 \leq i \leq c} \mathcal{O}_{\mathcal{P}_N}(-\sum_{j=1}^c m_j + m_i) \stackrel{\alpha_{i-1}}{\longrightarrow} \bigoplus_{1 \leq i \leq c} \mathcal{O}_{\mathcal{P}_N}(-m_i) \longrightarrow 0.$$ (4.8)

Multiplying this exact sequence with the local free sheaf $\Omega_{\mathcal{P}_N}^r(\log D^*)$ yields the assertion.

**Theorem 4.5.** Under the assumption $1 \leq q < n - \text{depth} T - \mu$ one gets

$$H^q(Y, \Omega_{\mathcal{P}_N}^r(\log D)(t)) = 0 \text{ for all } t \in \mathbb{Z}.$$
Proof. We write instead of (4.8) short exact sequences and obtain

\[0 \to \Omega_{\mathbb{P}^N}^r(\log D^*)(-\sum_{i=1}^c m_i) \to \bigoplus_{1 \leq i \leq c} \Omega_{\mathbb{P}^N}^r(\log D^*)(-\sum_{j=1}^c m_j + m_i) \to \operatorname{Im} \alpha_{c-1} \to 0\]

\[0 \to \operatorname{Im} \alpha_{c-1} \to \bigoplus_{1 \leq i < i_2 \leq c} \Omega_{\mathbb{P}^N}^r(\log D^*)(-\sum_{j=1}^c m_j + m_{i_1} + m_{i_2}) \to \operatorname{Im} \alpha_{c-2} \to 0\]

\[\vdots\]

\[0 \to \operatorname{Im} \alpha_2 \to \bigoplus_{1 \leq i \leq c} \Omega_{\mathbb{P}^N}^r(\log D^*)(-m_i) \to \operatorname{Im} \alpha_1 \to 0\]

\[0 \to \operatorname{Im} \alpha_1 \to \Omega_{\mathbb{P}^N}^r(\log D^*) \to \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega_{\mathbb{P}^N}^r(\log D^*) \to 0.\]

Using the long exact cohomology sequences yields a vanishing criterion for \(H^q(Y, \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega_{\mathbb{P}^N}^r(\log D^*)(t))\). We have

\[H^q(Y, \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega_{\mathbb{P}^N}^r(\log D^*)(t)) = 0 \text{ if } H^q(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^r(\log D^*)(t)) = 0 \text{ and } H^{q+1}(\mathbb{P}^N, \operatorname{Im} \alpha_1(t)) = 0\]

\[H^{q+1}(\mathbb{P}^N, \operatorname{Im} \alpha_1(t)) = 0 \text{ if } H^{q+1}(\mathbb{P}^N, \bigoplus_{1 \leq i \leq c} \Omega_{\mathbb{P}^N}^r(\log D^*)(t - m_i)) = 0 \text{ and } H^{q+2}(\mathbb{P}^N, \operatorname{Im} \alpha_2(t)) = 0\]

\[\vdots\]

\[H^{q+c-1}(\mathbb{P}^N, \operatorname{Im} \alpha_{c-1}(t)) = 0 \text{ if } H^{q+c-1}(\mathbb{P}^N, \bigoplus_{1 \leq i \leq c} \Omega_{\mathbb{P}^N}^r(\log D^*)(t - \sum_{j=1}^c m_j + m_i)) = 0 \text{ and } H^{q+c}(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^r(\log D^*)(t - \sum_{i=1}^c m_i)) = 0.\]

Applying Theorem 3.6 (d) yields \(H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega_{\mathbb{P}^N}^r(\log D^*)) = 0\) for \(1 \leq q < n\)–depth \(T^\prime\). Now we study \(H^q(Y, \Omega_{\mathcal{L}}^r(\log D)(t))\) with the aid of (4.6). Decomposing (4.6) in short exact sequences delivers

\[0 \to E_T^0 \to E_T^{\mu-1} \to \operatorname{Im} \beta_{\mu-1} \to 0\]

\[0 \to \operatorname{Im} \beta_{\mu-1} \to E_T^{\mu-2} \to \operatorname{Im} \beta_{\mu-2} \to 0\]

\[\vdots\]

\[0 \to \operatorname{Im} \beta_2 \to E_T^1 \to \operatorname{Im} \beta_1 \to 0\]

\[0 \to \operatorname{Im} \beta_1 \to \Omega_{\mathbb{P}^N|Y}^r(\log D^*) \to \Omega_Y^r(\log D) \to 0.\]
With \( E_\tau^*(t) = \bigoplus_{A \in \mathcal{M}_\tau(T)} \Omega_{\mathbb{P}^N|Y}^T(\log D^*)(t + t(A)) \) one has

\[
H^q(Y, \Omega_Y^T(\log D)(t)) = 0 \text{ if } H^q(Y, \Omega_{\mathbb{P}^N|Y}^T(\log D^*)(t)) = 0 \quad \text{and } H^{q+1}(Y, \operatorname{Im} \beta(t)) = 0
\]

\[
\vdots
\]

\[
H^{q+\mu-1}(Y, \operatorname{Im} \beta_{\mu-1}(t)) = 0 \text{ if } H^{q+\mu-1}(Y, E_{\tau}^{\mu-1}(t)) = 0 \quad \text{and } H^{q+\mu}(Y, E_{\tau}^\mu(t)) = 0.
\]

This implies \( H^q(Y, \Omega_Y^T(\log D)(t)) = 0 \) for \( 1 \leq q < n - \text{depth} \, T - \mu. \)

Now assume for instance \( \mu < n - \text{depth} \, T. \) Then for each \( t \in \mathbb{Z} \) it follows from our exact sequences: \( H^q(\mathbb{P}^N, \operatorname{Im} \alpha_i(t)) = 0 \) if \( 1 \leq q \leq \mu + i \),

\( H^q(Y, \mathcal{O}_Y(t) \otimes \mathcal{O}_{\mathbb{P}^N}^T(\log D^*))(0) = 0 \) if \( 1 \leq q \leq \mu + c \),

\( H^q(Y, E^j_{\tau}(t)) = 0 \) if \( 1 \leq q \leq j \), \( H^q(Y, \operatorname{Im} \beta_j(t)) = 0 \) if \( 1 \leq q \leq j \).

In particular, the cohomology groups \( H^1(\ldots) \) of all these sheaves vanish. Therefore, we have the opportunity to calculate the dimensions of their cohomology groups \( H^0(\ldots): \)

\[ h^T(t) := h^T(t) + \sum_{s=1}^c (-1)^s \cdot \sum_{1 \leq i_1 < i_2 < \ldots < i_s \leq c} h^T(t - m_{i_1} - m_{i_2} - \ldots - m_{i_s}). \]

Because of (4.8) we have \( \dim H^0(Y, \mathcal{O}_Y(t) \otimes \mathcal{O}_{\mathbb{P}^N}^T(\log D^*)(t)) = h^T_{(m)}(t) \) and using (4.6) we get the following formula:

**Theorem 4.6.** If \( \mu < \dim Y - \text{depth} \, T \) then

\[
\dim H^0(Y, \Omega_Y^T(\log D)(t)) = \sum_{A \in \mathcal{M}_\tau(T)} (-1)^{t - g(A)} \cdot h^T_{(m)}(t + t(A))
\]

with \( t(A) = \sum_{i=1}^c (g_{i+1}(A) - g_i(A)) \cdot m_i. \)

In particular for \( t = 0 : H^0(Y, \Omega_Y^T(\log D)) = 0 \) if \( \mu < \dim Y - \text{depth} \, T. \)

**Remark 4.7.** For regular \( T \)-symmetrical tensor differential forms one has \( H^0(Y, \Omega_Y^T) = 0 \) if \( \mu < \dim Y \).

### 4.3 Symmetric Differential Forms

We consider symmetrical differential forms with logarithmic poles as a special case, that means, \( T \) is a Young tableau with \( r \) boxes and only one row (depth \( T = 1, l = \text{length} \, T = r \)). Let \( D^* = H \) be the prime divisor on projective space \( \mathbb{P}^N \) and let \( D \) be the prime divisor on the \( n \)-dimensional complete intersection \( Y \) as above \( (n \geq 2) \). Distinguishing the cases \( r \leq c \) and \( c < r \) we obtain two exact sequences as symmetrical power of (4.7):

Assume at first
Assume Theorem 4.8. In the case $c < r$ we have Lemma 4.4 with the sheaf

$$\sum_{1 \leq i_1 < \ldots < i_r \leq c} O_Y(-m_{i_1} - m_{i_2} - \ldots - m_{i_r}) \to \cdots$$

Furthermore, we have Theorem 4.5 with the sheaf

$$\sum_{1 \leq i_1 < \ldots < i_{r-1} \leq c} O_Y(-m_{i_1} - m_{i_2} - \ldots - m_{i_{r-1}}) \otimes \mathscr{O}_{\mathbb{P}^N} \Omega^1_{\mathbb{P}^N}(\log D^*) \to \cdots$$

$$\to O_Y \otimes \mathscr{O}_{\mathbb{P}^N} S^r \Omega^1_{\mathbb{P}^N}(\log D^*) \to S^r \Omega^1_{\mathbb{P}^N}(\log D) \to 0$$

In the case $c = r$ the following sequence is exact:

$$0 \to O_Y(-\sum_{j=1}^c m_j) \otimes O_{\mathbb{P}^N} \Omega^r_{\mathbb{P}^N}(\log D^*) \to \cdots$$

$$\to \bigoplus_{1 \leq i_1 < \ldots < i_r \leq c} O_Y(-m_{i_1} + m_{i_2} + \ldots + m_{i_r}) \otimes O_{\mathbb{P}^N} \Omega^{r-1}_{\mathbb{P}^N}(\log D^*) \to \cdots$$

$$\to O_Y \otimes O_{\mathbb{P}^N} S^r \Omega^1_{\mathbb{P}^N}(\log D^*) \to S^r \Omega^1_{\mathbb{P}^N}(\log D) \to 0$$

Furthermore, we have Lemma 4.4 with the sheaf $S^r \Omega^1_{\mathbb{P}^N}(\log D^*)$ instead of $\Omega^r_{\mathbb{P}^N}(\log D^*)$.

With the corresponding cohomology sequences we get:

**Theorem 4.8.** Assume $n = \dim Y \geq 2$.

(a) If $1 \leq q \leq n - 2$ then $H^q(Y, O_Y(t) \otimes \mathscr{O}_{\mathbb{P}^N} S^r \Omega^1_{\mathbb{P}^N}(\log D^*)) = 0 \forall t \in \mathbb{Z}$

(b) $\dim H^0(Y, O_Y(t) \otimes \mathscr{O}_{\mathbb{P}^N} S^r \Omega^1_{\mathbb{P}^N}(\log D^*)) = \dim H^0(\mathbb{P}^N, S^r \Omega^1_{\mathbb{P}^N}(\log D^*)|t) + \sum_{j=1}^c (-1)^j \cdot \sum_{1 \leq i_1 < \ldots < i_{r-1} \leq c} \dim H^0(\mathbb{P}^N, S^r \Omega^1_{\mathbb{P}^N}(\log D^*)(t - m_{i_1} - \ldots - m_{i_r}))$

(c) $H^0(Y, O_Y(t) \otimes \mathscr{O}_{\mathbb{P}^N} S^r \Omega^1_{\mathbb{P}^N}(\log D^*)) \neq 0$ if $t \geq r$

(d) In case $t = 0$: $H^0(Y, O_Y \otimes \mathscr{O}_{\mathbb{P}^N} S^r \Omega^1_{\mathbb{P}^N}(\log D^*)) = 0$ for all $r > 0$

**Theorem 4.9.**

(a) If $r \leq c$ and $1 \leq q < n - r$ then $H^q(Y, S^r \Omega^1_Y(\log D)(t)) = 0 \forall t \in \mathbb{Z}$

(b) If $c < r$ and $1 \leq q < n - c - 1$ then $H^q(Y, S^r \Omega^1_Y(\log D)(t)) = 0 \forall t \in \mathbb{Z}$

**Proof.** By Theorem 4.5 we know $H^q(Y, \Omega^r_Y(\log D)(t)) = 0$ for all $t \in \mathbb{Z}$ if $1 \leq q < n - \text{depth } T - \mu$. For symmetric differential forms we have depth $T = 1$ and $\mu = \sum_{i=1}^c d_i = \min\{c, r\}$, where $d_i = 1$ for $i \leq r$ and $d_i = 0$ for $i > r$. This proves (b). Under condition $r \leq c$ one gets the stronger result (a) since $H^q(Y, O_Y(t)) = 0$ for $1 \leq q < n$ and for all $t \in \mathbb{Z}$. \(\square\)

**Theorem 4.10.**
(c) If \( r \leq c \) and \( r < n \) then
\[
H^0(Y, S^r \Omega_Y^1 (\log D))(t) = \dim H^0(Y, \mathcal{O}_Y(t) \otimes \mathcal{O}_{\mathbb{P}^N} S^r \Omega_{SN}^1 (\log D^*)) + 
\sum_{k=1}^{r-1} (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq c} \dim H^0(Y, \mathcal{O}_Y(t - \sum_{j=1}^k m_{i_j}) \otimes \mathcal{O}_{\mathbb{P}^N} S^{r-k} \Omega_{SN}^1 (\log D^*))
\]
\[
+ (-1)^r \sum_{1 \leq i_1 < \ldots < i_r \leq c} \dim H^0(Y, \mathcal{O}_Y(t - m_{i_1} - \ldots - m_{i_r})
\]

(d) If \( c < r \) and \( c < n - 1 \) then
\[
H^0(Y, S^r \Omega_Y^1 (\log D))(t) = \dim H^0(Y, \mathcal{O}_Y(t) \otimes \mathcal{O}_{\mathbb{P}^N} S^r \Omega_{SN}^1 (\log D^*)) + 
\sum_{k=1}^{c} (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq c} \dim H^0(Y, \mathcal{O}_Y(t - \sum_{j=1}^k m_{i_j}) \otimes \mathcal{O}_{\mathbb{P}^N} S^{r-k} \Omega_{SN}^1 (\log D^*))
\]

Proof. Statements (c) and (d) follow from the related exact sequences since under these premises by Theorem 4.8 the cohomology groups \( H^1(\ldots) \) of all these sheaves vanish (cf. Theorem 4.8 and Theorem 3.7).

Finally, it is easy to see:

Theorem 4.11.

(e) If \( t < r \leq \min(c, n - 1) \) then \( H^0(Y, S^r \Omega_Y^1 (\log D))(t) = 0 \).

(f) If \( t < r \) and \( c < \min(r, n - 1) \) then \( H^0(Y, S^r \Omega_Y^1 (\log D))(t) = 0 \).

(g) If \( c < n - 1 \) then \( H^0(Y, S^r \Omega_Y^1 (\log D)) = 0 \) for all \( r > 0 \).

(h) If \( 0 < r < n \) then \( H^0(Y, S^r \Omega_Y^1 (\log D)) = 0 \).

Remark 4.12. On the other hand, for regular symmetrical differential forms on complete intersections it is well known:
If \( c < n \) then \( H^0(Y, S^r \Omega_Y^1) = 0 \) for all \( r > 0 \).

References
