

A PRIORI BOUNDS FOR WEAK SOLUTIONS TO ELLIPTIC EQUATIONS WITH NONSTANDARD GROWTH

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ABSTRACT. In this paper we study elliptic equations with a nonlinear conormal derivative boundary condition involving nonstandard growth terms. By means of the localization method and De Giorgi's iteration technique we derive global a priori bounds for weak solutions of such problems.

1. INTRODUCTION

The present paper is concerned with global a priori bounds for elliptic equations with nonlinear conormal derivative boundary conditions which may contain nonlinearities with variable growth exponents. More precisely, let Ω be a bounded domain in \mathbb{R}^N , $N > 1$, with Lipschitz boundary $\Gamma := \partial\Omega$ and let $p \in C(\overline{\Omega})$ be a function that satisfies $1 < p^- := \inf_{\overline{\Omega}} p(x)$. We deal with elliptic equations of the form

$$\begin{aligned} -\operatorname{div} \mathcal{A}(x, u, \nabla u) &= \mathcal{B}(x, u, \nabla u) && \text{in } \Omega, \\ \mathcal{A}(x, u, \nabla u) \cdot \nu &= \mathcal{C}(x, u) && \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \Gamma$, and \mathcal{A}, \mathcal{B} and \mathcal{C} satisfy suitable $p(x)$ -structure conditions, see (H) below.

An important special case of (1.1) which fits in our setting is given by

$$-\Delta_{p(x)} u = \mathcal{B}(x, u, \nabla u) \quad \text{in } \Omega, \quad |\nabla u|^{p(x)-2} \partial_\nu u = \mathcal{C}(x, u) \quad \text{on } \Gamma.$$

Here the operator $\operatorname{div} \mathcal{A}$ becomes the so-called $p(x)$ -Laplacian

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u),$$

which reduces to the standard p -Laplacian if $p(x) \equiv p$.

In recent years there has been a growing interest in the study of elliptic problems with a $p(x)$ -structure, which are also termed problems with nonstandard growth conditions. Equations of this type appear in the study of non-Newtonian fluids with thermo-convective effects (see Antontsev and Rodrigues [4], Zhikov [35]), electro-rheological fluids (see Diening [8], Rajagopal and Růžička [29], Růžička [31]), the thermistor problem (see Zhikov [36]), or the problem of image recovery (see Chen et al. [5]).

Throughout the paper we impose the following conditions.

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(H) The functions $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, and $\mathcal{C} : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions satisfying the subsequent structure conditions:

$$(H1) \quad |\mathcal{A}(x, s, \xi)| \leq a_0 |\xi|^{p(x)-1} + a_1 |s|^{q_0(x) \frac{p(x)-1}{p(x)}} + a_2, \quad \text{for a.a. } x \in \Omega,$$

$$(H2) \quad \mathcal{A}(x, s, \xi) \cdot \xi \geq a_3 |\xi|^{p(x)} - a_4 |s|^{q_0(x)} - a_5, \quad \text{for a.a. } x \in \Omega,$$

$$(H3) \quad |\mathcal{B}(x, s, \xi)| \leq b_0 |\xi|^{q_0(x) \frac{q_0(x)-1}{q_0(x)}} + b_1 |s|^{q_0(x)-1} + b_2, \quad \text{for a.a. } x \in \Omega,$$

$$(H4) \quad |\mathcal{C}(x, s)| \leq c_0 |s|^{q_1(x)-1} + c_1, \quad \text{for a.a. } x \in \Gamma,$$

and for all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$. Here a_i, b_j and c_l are positive constants, $p \in C(\overline{\Omega})$ with $\inf_{\overline{\Omega}} p(x) > 1$, and $q_0 \in C(\overline{\Omega})$ as well as $q_1 \in C(\Gamma)$ are chosen such that

$$p(x) \leq q_0(x) < p^*(x), \quad x \in \overline{\Omega}, \quad \text{and} \quad p(x) \leq q_1(x) < p_*(x), \quad x \in \Gamma,$$

with the critical exponents

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases} \quad p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

A function $u \in W^{1,p(\cdot)}(\Omega)$ is said to be a **weak solution** (**subsolution**, **supersolution**) of equation (1.1) if

$$\int_{\Omega} \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi dx = (\leq, \geq) \int_{\Omega} \mathcal{B}(x, u, \nabla u) \varphi dx + \int_{\Gamma} \mathcal{C}(x, u) \varphi d\sigma, \quad (1.2)$$

holds for all nonnegative test functions $\varphi \in W^{1,p(\cdot)}(\Omega)$, where $d\sigma$ denotes the usual $(N-1)$ -dimensional surface measure.

This definition makes sense, since thanks to assumption (H) the integrals in (1.2) are finite, by Hölder's inequality and embedding results for $W^{1,p(\cdot)}(\Omega)$ -functions, see below.

The main goal of this paper is to prove a priori bounds for weak sub- and supersolutions, in particular for weak solutions of problem (1.1). Using the notation $y_+ = \max(y, 0)$, our main result reads as follows.

Theorem 1.1. *Let the assumptions in (H) be satisfied. Then there exist positive constants $\alpha = \alpha(p, q_0, q_1)$ and $C = C(p, q_0, q_1, a_3, a_4, a_5, b_0, b_1, b_2, c_0, c_1, N, \Omega)$ such that the following assertions hold.*

(i) *If $u \in W^{1,p(\cdot)}(\Omega)$ is a weak subsolution of (1.1) then*

$$\operatorname{ess\,sup}_{\Omega} u \leq 2 \max \left(1, C \left[\int_{\Omega} u_+^{q_0(x)} dx + \int_{\Gamma} u_+^{q_1(x)} d\sigma \right]^{\alpha} \right).$$

(ii) *If $u \in W^{1,p(\cdot)}(\Omega)$ is a weak supersolution of (1.1) then*

$$\operatorname{ess\,inf}_{\Omega} u \geq -2 \max \left(1, C \left[\int_{\Omega} (-u)_+^{q_0(x)} dx + \int_{\Gamma} (-u)_+^{q_1(x)} d\sigma \right]^{\alpha} \right).$$

Note that the constants a_0, a_1, a_2 , which appear in (H1), do not play any role in determining the constants α and C . The finiteness of the right-hand sides in (i) and (ii) is a consequence of the compact embedding $W^{1,p(\cdot)}(\Omega) \rightarrow L^{q_0(\cdot)}(\Omega)$ and the fact that the trace operator is a bounded operator from $W^{1,p(\cdot)}(\Omega)$ into $L^{q_1(\cdot)}(\Gamma)$ (see Fan et al. [15, Theorem 1.3] and Fan [12, Corollary 2.4]). We further point out that we merely assume continuity for the variable exponents p, q_0 , and q_1 ; log-Hölder continuity conditions are not required.

Our proof of Theorem 1.1 uses De Giorgi’s iteration technique and the localization method. By means of the latter we are able to reduce the estimates involving variable exponents to ones with constant exponents, which then also allows us to apply classical embedding results. This crucial step in the proof is achieved by means of an appropriate partition of unity.

By the definition of sub- and supersolution of (1.1) one easily verifies that a weak solution is both a weak subsolution and a weak supersolution. Hence we have the following.

Corollary 1.2. *Let the assumptions (H) be satisfied and let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak solution of (1.1). Then $u \in L^\infty(\Omega)$ and the estimates in (i) and (ii) from Theorem 1.1 are valid.*

The main novelty of the paper consists in the generality of the assumptions needed to establish the boundedness of weak solutions to (1.1). In particular the assumptions on the nonlinearity \mathcal{C} are rather general, allowing for a growth term with variable exponent, which seems to be optimal. Another novelty is the use of the localization technique in the context of global a priori estimates for problems with variable exponents and nonlinear conormal derivative boundary conditions.

Let us comment on some relevant known results on elliptic problems with $p(x)$ -structure. Local boundedness of solutions to the equation

$$-\operatorname{div} \mathcal{A}(x, u, \nabla u) = \mathcal{B}(x, u, \nabla u) \quad \text{in } \Omega, \tag{1.3}$$

has been studied by Fan and Zhao [16]. There it is shown that under suitable structure conditions every weak solution u of (1.3) (corresponding to test functions $\varphi \in W_0^{1,p(x)}(\Omega)$) belongs to $L^\infty_{\text{loc}}(\Omega)$, and if in addition u is bounded on the boundary Γ , then $u \in L^\infty(\Omega)$. The proof uses De Giorgi iterations as well. Recently, Gasiński and Papageorgiou (see [19, Proposition 3.1]) studied global a priori bounds for weak solutions to the equation

$$\begin{aligned} -\Delta_{p(x)} u &= g(x, u) && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \Gamma, \end{aligned} \tag{1.4}$$

where the Carathéodory function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies a subcritical growth condition. They proved that every weak solution $u \in W^{1,p(\cdot)}(\Omega)$ of problem (1.4) belongs to $L^\infty(\Omega)$ provided $p \in C^1(\bar{\Omega})$ satisfying $1 < \min_{x \in \bar{\Omega}} p(x)$.

L^∞ -estimates for solutions of (1.1) in case $p(x) \equiv p$ with $q_0(x) = q_1(x) \equiv p$ have been established by the first author in [33, 34] following Moser’s iteration technique (for constant p see also Pucci and Servadei [28]).

Concerning boundedness and regularity results for problems of type (1.3), in particular for the special case

$$-\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = 0,$$

we further refer to Acerbi and Mingione [1, 2], Antontsev and Consiglieri [3], Chiadò Piat and Coscia [6], Diening et al. [9], Eleuteri and Habermann [11], Fan [13, 14], Fan and Zhao [18], Habermann and Zatorska-Goldstein [20], Harjulehto et al. [21], Liskevich and Skrypnik [25], Lukkari [26, 27] and the references given therein.

The paper is organized as follows. In Section 2 we fix some notation and recall the definition of the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. We further state a lemma on sequences of numbers which will be needed for the De Giorgi

iterations. The main result is proved in Section 3. From the structure conditions we first derive truncated energy estimates. These are then used, together with the localization method and embedding results, to prove suitable iterative inequalities, which in turn imply the desired a priori bounds.

2. NOTATIONS AND PRELIMINARIES

Suppose that Ω is a bounded domain in \mathbb{R}^N with Lipschitz boundary Γ and let $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$. We set $p^- := \min_{x \in \overline{\Omega}} p(x)$ and $p^+ := \max_{x \in \overline{\Omega}} p(x)$, then $p^- > 1$ and $p^+ < \infty$. By $L^{p(\cdot)}(\Omega)$ we identify the variable exponent Lebesgue space which is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \mid u : \Omega \rightarrow \mathbb{R} \text{ is measurable and } \int_{\Omega} |u|^{p(x)} dx < +\infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \tau > 0 : \int_{\Omega} \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)}.$$

For more information and basic properties of variable exponent spaces we refer the reader to the papers of Fan and Zhao [17], Kováčik and Rákosník [22] and the recent monograph of Diening et al. [10]. If $p(x) \equiv p$ is a constant, the usual Sobolev space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |u|^p dx \right)^{\frac{1}{p}}.$$

For $q_0 \in C(\overline{\Omega})$ and $q_1 \in C(\Gamma)$ (as in (H)) we define

$$\begin{aligned} q_0^+ &= \max_{\overline{\Omega}} q_0(x), & q_0^- &= \min_{\overline{\Omega}} q_0(x), \\ q_1^+ &= \max_{\Gamma} q_1(x), & q_1^- &= \min_{\Gamma} q_1(x). \end{aligned}$$

For $s \in [1, \infty)$ we further use the notation

$$s^* = \begin{cases} \frac{Ns}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \geq N, \end{cases} \quad s_* = \begin{cases} \frac{(N-1)s}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \geq N. \end{cases}$$

The following lemma concerning the geometric convergence of sequences of numbers will be needed for the De Giorgi iteration arguments below. It can be found, for example in [32]. The case $\delta_1 = \delta_2$ is contained in [23, Chapter II, Lemma 5.6], see also [7, Chapter I, Lemma 4.1].

Lemma 2.1. *Let $\{Y_n\}, n = 0, 1, 2, \dots$, be a sequence of positive numbers, satisfying the recursion inequality*

$$Y_{n+1} \leq Kb^n (Y_n^{1+\delta_1} + Y_n^{1+\delta_2}), \quad n = 0, 1, 2, \dots,$$

for some $b > 1$, $K > 0$, and $\delta_2 \geq \delta_1 > 0$. If

$$Y_0 \leq (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}},$$

then

$$Y_n \leq (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}} b^{-\frac{n}{\delta_1}}, \quad n \in \mathbb{N},$$

in particular $\{Y_n\} \rightarrow 0$ as $n \rightarrow \infty$.

3. TRUNCATED ENERGY ESTIMATES AND PROOF OF THEOREM 1.1

Our proof of the sup-bounds for weak subsolutions of (1.1) is based on the following lemma on truncated energy estimates.

Lemma 3.1. *Let the conditions in (H) be satisfied. If u is a weak subsolution of (1.1), there holds*

$$\int_{A_k} |\nabla u|^{p(x)} dx \leq d_1 \int_{A_k} u^{q_0(x)} dx + d_2 \int_{\Gamma_k} u^{q_1(x)} d\sigma,$$

where

$$A_k = \{x \in \Omega : u(x) > k\}, \quad \Gamma_k = \{x \in \Gamma : u(x) > k\}, \quad k \geq 1,$$

and $d_1 = 2a_3^{-1}(a_4 + a_5 + b_1 + b_2 + b_0\varepsilon^{-(q_0^+ + 1)})$, $d_2 = 2a_3^{-1}(c_0 + c_1)$, and $\varepsilon = \min(1, \frac{a_3}{2b_0})$.

Proof. Let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak subsolution of (1.1) and let $k \geq 1$. Taking $\varphi = (u - k)_+ = \max(u - k, 0) \in W^{1,p(\cdot)}(\Omega)$ (see [24, Lemma 3.2]) as test function in (1.2) with the ' \leq '-sign we obtain

$$\begin{aligned} & \int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla(u - k) dx \\ & \leq \int_{A_k} \mathcal{B}(x, u, \nabla u)(u - k) dx + \int_{\Gamma_k} \mathcal{C}(x, u)(u - k) d\sigma. \end{aligned} \quad (3.1)$$

Using the structure condition (H2) we estimate the left-hand side of (3.1) as follows.

$$\begin{aligned} & \int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla(u - k) dx \\ & = \int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla u dx \\ & \geq \int_{A_k} \left(a_3 |\nabla u|^{p(x)} - a_4 |u|^{q_0(x)} - a_5 \right) dx \\ & \geq a_3 \int_{A_k} |\nabla u|^{p(x)} dx - (a_4 + a_5) \int_{A_k} |u|^{q_0(x)} dx, \end{aligned} \quad (3.2)$$

as $u^{q_0(x)} > u > 1$ in A_k . Now, we are going to estimate the right-hand side of (3.1). By Young's inequality with $\varepsilon \in (0, 1]$ and condition (H3) we have

$$\begin{aligned}
& \int_{A_k} \mathcal{B}(x, u, \nabla u)(u - k) dx \\
& \leq \int_{A_k} \left[b_0 |\nabla u|^{p(x) \frac{q_0(x)-1}{q_0(x)}} + b_1 |u|^{q_0(x)-1} + b_2 \right] (u - k) dx \\
& \leq b_0 \int_{A_k} \left[\varepsilon^{\frac{q_0(x)-1}{q_0(x)}} |\nabla u|^{p(x) \frac{q_0(x)-1}{q_0(x)}} \varepsilon^{-\frac{q_0(x)-1}{q_0(x)}} u \right] dx + (b_1 + b_2) \int_{A_k} |u|^{q_0(x)} dx \quad (3.3) \\
& \leq b_0 \int_{A_k} \varepsilon |\nabla u|^{p(x)} dx + b_0 \int_{A_k} \varepsilon^{-(q_0(x)-1)} u^{q_0(x)} dx + (b_1 + b_2) \int_{A_k} |u|^{q_0(x)} dx \\
& \leq \varepsilon b_0 \int_{A_k} |\nabla u|^{p(x)} dx + \left(b_0 \varepsilon^{-(q_0^+-1)} + b_1 + b_2 \right) \int_{A_k} u^{q_0(x)} dx.
\end{aligned}$$

Thanks to condition (H4), the boundary integral can be estimated through

$$\begin{aligned}
\int_{\Gamma_k} \mathcal{C}(x, u)(u - k) d\sigma & \leq \int_{\Gamma_k} (c_0 |u|^{q_1(x)-1} + c_1)(u - k) d\sigma \\
& \leq (c_0 + c_1) \int_{\Gamma_k} u^{q_1(x)} d\sigma, \quad (3.4)
\end{aligned}$$

as $u > 1$ on Γ_k . Combining (3.1)–(3.4) and choosing $\varepsilon = \min(1, \frac{a_3}{2b_0})$ gives

$$\begin{aligned}
& \frac{a_3}{2} \int_{A_k} |\nabla u|^{p(x)} dx \\
& \leq \left(a_4 + a_5 + b_1 + b_2 + b_0 \varepsilon^{-(q_0^+-1)} \right) \int_{A_k} u^{q_0(x)} dx + (c_0 + c_1) \int_{\Gamma_k} u^{q_1(x)} d\sigma.
\end{aligned}$$

Dividing the last inequality by $\frac{a_3}{2} > 0$ yields the assertion of the lemma. \square

The corresponding result for supersolutions reads as follows.

Lemma 3.2. *Let the conditions in (H) be satisfied. If u is a weak supersolution of (1.1), there holds*

$$\int_{\tilde{A}_k} |\nabla u|^{p(x)} dx \leq d_1 \int_{\tilde{A}_k} (-u)^{q_0(x)} dx + d_2 \int_{\tilde{\Gamma}_k} (-u)^{q_1(x)} d\sigma,$$

where

$$\tilde{A}_k = \{x \in \Omega : -u(x) > k\}, \quad \tilde{\Gamma}_k = \{x \in \Gamma : -u(x) > k\}, \quad k \geq 1,$$

and d_1 and d_2 are the same constants as in Lemma 3.1.

Proof. The proof is analogous to the previous one. We take $\varphi = -(u + k)_- = -\min(u + k, 0) \geq 0$ as test function in (1.2), which now holds with the ' \geq '-sign, and use the same arguments as in the proof of Lemma 3.1. This yields the asserted inequality. \square

Now we are in position to prove the main result of this paper.

Proof of Theorem 1.1.

(i) **Definition of the iteration variables Z_n , \tilde{Z}_n , and basic estimates.**

Let now

$$k_n = k \left(2 - \frac{1}{2^n} \right), \quad n = 0, 1, 2, \dots,$$

with $k \geq 1$ specified later and put

$$Z_n := \int_{A_{k_n}} (u - k_n)^{q_0(x)} dx, \quad \tilde{Z}_n := \int_{\Gamma_{k_n}} (u - k_n)^{q_1(x)} d\sigma.$$

We have

$$\begin{aligned} Z_n &\geq \int_{A_{k_{n+1}}} (u - k_n)^{q_0(x)} dx \geq \int_{A_{k_{n+1}}} u^{q_0(x)} \left(1 - \frac{k_n}{k_{n+1}} \right)^{q_0(x)} dx \\ &\geq \int_{A_{k_{n+1}}} \frac{1}{2^{q_0(x)(n+2)}} u^{q_0(x)} dx, \end{aligned}$$

and thus

$$\int_{A_{k_{n+1}}} u^{q_0(x)} dx \leq 2^{q_0^+(n+2)} Z_n. \quad (3.5)$$

Analogously, we see that

$$\int_{\Gamma_{k_{n+1}}} u^{q_1(x)} d\sigma \leq 2^{q_1^+(n+2)} \tilde{Z}_n. \quad (3.6)$$

From (3.5)–(3.6) and Lemma 3.1 with k being replaced by $k_{n+1} \geq 1$ it follows that

$$\int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p(x)} dx \leq d_3 a^n (Z_n + \tilde{Z}_n), \quad (3.7)$$

where $d_3 = \max(d_1 2^{2q_0^+}, d_2 2^{2q_1^+})$ and $a = \max(2^{q_0^+}, 2^{q_1^+})$.

Furthermore, we have

$$\begin{aligned} |A_{k_{n+1}}| &\leq \int_{A_{k_{n+1}}} \left(\frac{u - k_n}{k_{n+1} - k_n} \right)^{q_0(x)} dx \\ &\leq \int_{A_{k_n}} \frac{2^{q_0(x)(n+1)}}{k^{q_0(x)}} (u - k_n)^{q_0(x)} dx \\ &\leq \frac{2^{q_0^+(n+1)}}{k^{q_0^-}} \int_{A_{k_n}} (u - k_n)^{q_0(x)} dx \\ &= \frac{2^{q_0^+(n+1)}}{k^{q_0^-}} Z_n. \end{aligned} \quad (3.8)$$

(ii) **Partition of unity.** By compactness of $\bar{\Omega}$, for any $R > 0$ there exists a finite open cover $\{B_i(R)\}_{i=1, \dots, m}$ of balls $B_i := B_i(R)$ with radius R such that $\bar{\Omega} \subset \bigcup_{i=1}^m B_i(R)$. Moreover, since $p \in C(\bar{\Omega})$, $q_0 \in C(\bar{\Omega})$, and $q_1 \in C(\Gamma)$, these functions are uniformly continuous on $\bar{\Omega}$ and Γ , respectively. Recalling that

$$p(x) \leq q_0(x) < p^*(x), \quad x \in \bar{\Omega}, \quad \text{and} \quad p(x) \leq q_1(x) < p_*(x), \quad x \in \Gamma,$$

we may take $R > 0$ small enough such that

$$p_i^+ \leq q_{0,i}^+ < (p_i^-)^*, \quad p_i^+ \leq q_{1,i}^+ < (p_i^-)_*, \quad i = 1, \dots, m,$$

where

$$\begin{aligned} p_i^+ &= \max_{B_i \cap \bar{\Omega}} p(x), & q_{0,i}^+ &= \max_{B_i \cap \bar{\Omega}} q_0(x), \\ p_i^- &= \min_{B_i \cap \bar{\Omega}} p(x), & q_{1,i}^+ &= \max_{B_i \cap \Gamma} q_1(x). \end{aligned}$$

We next choose a partition of unity $\{\xi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^N)$ associated to the open cover $\{B_i(R)\}_{i=1,\dots,m}$ (see e.g. [30, Thm. 6.20]), that is, we have

$$\text{supp } \xi_i \subset B_i, \quad 0 \leq \xi_i \leq 1, \quad i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^m \xi_i = 1 \quad \text{on } \bar{\Omega}.$$

Let $L > 0$ be a positive constant such that

$$|\nabla \xi_i| \leq L, \quad i = 1, \dots, m.$$

(iii) Estimating the gradient term in (3.7) from below. Using the partition of unity from step (ii) we have

$$\begin{aligned} & \int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p(x)} dx \\ &= \int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p(x)} \sum_{i=1}^m \xi_i dx \\ &\geq \sum_{i=1}^m \int_{A_{k_{n+1}}} (|\nabla(u - k_{n+1})|^{p_i^-} - 1) \xi_i dx \\ &\geq \sum_{i=1}^m \int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx - m|A_{k_{n+1}}|, \end{aligned} \tag{3.9}$$

as $\xi_i \geq \xi_i^{p_i^-}$. From (3.9) we trivially deduce that for all $i = 1, \dots, m$,

$$\int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p(x)} dx \geq \int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx - m|A_{k_{n+1}}|. \tag{3.10}$$

Combining (3.7) and (3.10) and using (3.8) yields

$$\int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx \leq d_4 a^n (Z_n + \tilde{Z}_n) \tag{3.11}$$

for any $i = 1, \dots, m$, with the positive constant $d_4 = d_3 + m2^{q_0^+}$.

(iv) Estimating Z_{n+1} . Next we want to derive a suitable estimate for the term Z_{n+1} from above. To this end, we make again use of the partition of unity

introduced in step (ii). We have

$$\begin{aligned}
 Z_{n+1} &= \int_{A_{k_{n+1}}} (u - k_{n+1})^{q_0(x)} dx \\
 &= \int_{A_{k_{n+1}}} (u - k_{n+1})^{q_0(x)} \left(\sum_{i=1}^m \xi_i \right)^{q_0^+} dx \\
 &\leq \int_{A_{k_{n+1}}} (u - k_{n+1})^{q_0(x)} m^{q_0^+} \sum_{i=1}^m \xi_i^{q_0^+} dx \\
 &\leq m^{q_0^+} \sum_{i=1}^m \int_{A_{k_{n+1}}} (u - k_{n+1})^{q_0(x)} \xi_i^{q_0^+} dx \\
 &\leq m^{q_0^+} \sum_{i=1}^m \left[\int_{A_{k_{n+1}}} (u - k_{n+1})^{q_{0,i}^+} \xi_i^{q_{0,i}^+} dx + \int_{A_{k_{n+1}}} (u - k_{n+1})^{q_{0,i}^-} \xi_i^{q_{0,i}^-} dx \right],
 \end{aligned} \tag{3.12}$$

where we have set $q_{0,i}^- = \min_{B_i \cap \bar{\Omega}} q_0(x)$. Observe that $p_i^- \leq q_{0,i}^- \leq q_{0,i}^+ < (p_i^-)^*$ for all $i = 1, \dots, m$.

Let now $i \in \{1, \dots, m\}$ be fixed, and suppose that $r \in \{q_{0,i}^-, q_{0,i}^+\}$. Then $p_i^- \leq r < (p_i^-)^*$ and $r \leq q^+$, where $q^+ = \max(q_0^+, q_1^+)$. By Hölder's inequality and the continuous embedding $W^{1,p_i^-}(\Omega) \hookrightarrow L^{(p_i^-)^*}(\Omega)$, we may estimate as follows.

$$\begin{aligned}
 &\int_{\Omega} (u - k_{n+1})_+^r \xi_i^r dx \\
 &\leq \left(\int_{\Omega} (u - k_{n+1})_+^{(p_i^-)^*} \xi_i^{(p_i^-)^*} dx \right)^{\frac{r}{(p_i^-)^*}} |A_{k_{n+1}}|^{1 - \frac{r}{(p_i^-)^*}} \\
 &\leq C^{q^+} \left(\int_{\Omega} \left[|\nabla[(u - k_{n+1})_+ \xi_i]|^{p_i^-} \right. \right. \\
 &\quad \left. \left. + |(u - k_{n+1})_+ \xi_i|^{p_i^-} \right] dx \right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{r}{(p_i^-)^*}},
 \end{aligned} \tag{3.13}$$

where $C = \max(1, C(p_1^-, N), \dots, C(p_m^-, N))$ with $C(p_j^-, N)$ being the embedding constant corresponding to the embedding $W^{1,p_j^-}(\Omega) \hookrightarrow L^{(p_j^-)^*}(\Omega)$, $j = 1, \dots, m$. Thus C is independent of i . A simple calculation shows that the right-hand side of (3.13) can be estimated further to obtain

$$\begin{aligned}
 &\int_{\Omega} (u - k_{n+1})_+^r \xi_i^r dx \\
 &\leq d_5 \left(\int_{\Omega} |\nabla(u - k_{n+1})_+|^{p_i^-} \xi_i^{p_i^-} dx \right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{r}{(p_i^-)^*}} \\
 &\quad + d_6 \left(\int_{A_{k_{n+1}}} u^{q_0(x)} dx \right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{r}{(p_i^-)^*}}.
 \end{aligned} \tag{3.14}$$

Here $d_5 = d_5(q^+, C)$ and $d_6 = d_6(p^+, q^+, C, L)$ are positive constants, where L is the constant introduced in step (ii).

Using (3.5), (3.11), (3.12), and (3.14) with $r = q_{0,i}^+$ and $r = q_{0,i}^-$, respectively, for $i = 1, \dots, m$, we get

$$\begin{aligned}
Z_{n+1} \leq m^{q_0^+} \sum_{i=1}^m & \left[d_5 \left(d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{q_{0,i}^+}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{q_{0,i}^+}{(p_i^-)^*}} \right. \\
& + d_6 \left(2^{q_0^+(n+2)} Z_n \right)^{\frac{q_{0,i}^+}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{q_{0,i}^+}{(p_i^-)^*}} \\
& + d_5 \left(d_4 a^n (Z_n + \tilde{Z}_n) dx \right)^{\frac{q_{0,i}^-}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{q_{0,i}^-}{(p_i^-)^*}} \\
& \left. + d_6 \left(2^{q_0^+(n+2)} Z_n \right)^{\frac{q_{0,i}^-}{p_i^-}} |A_{k_{n+1}}|^{1 - \frac{q_{0,i}^-}{(p_i^-)^*}} \right]. \tag{3.15}
\end{aligned}$$

Setting

$$Y_n := Z_n + \tilde{Z}_n$$

and $\eta = \max \left(\frac{q_{0,1}^+}{(p_1^-)^*}, \dots, \frac{q_{0,m}^+}{(p_m^-)^*} \right)$ we have for $r \in \{q_{0,i}^+, q_{0,i}^-\}$

$$\begin{aligned}
\left(d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{r}{p_i^-}} & \leq (d_4)^{\frac{q^+}{p^-}} \left(a^{\frac{q^+}{p^-}} \right)^n (Y_n + Y_n^{\frac{q^+}{p^-}}), \\
\left(2^{q_0^+(n+2)} Z_n \right)^{\frac{r}{p_i^-}} & \leq \left(2^{\frac{(q^+)^2(n+2)}{p^-}} \right) (Y_n + Y_n^{\frac{q^+}{p^-}}), \\
|A_{k_{n+1}}|^{1 - \frac{r}{(p_i^-)^*}} & \leq 2^{q^+(n+1)} \left(\frac{1}{k^{q_0^-}} \right)^{1-\eta} (Y_n + Y_n^{1-\eta}). \tag{3.16}
\end{aligned}$$

Using these estimates, we conclude from (3.15) that

$$Z_{n+1} \leq d_7 d_8^n \frac{1}{k^{q_0^-(1-\eta)}} \left(Y_n^2 + Y_n^{2-\eta} + Y_n^{1+\frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\eta} \right) \tag{3.17}$$

where d_7 and d_8 are positive constants that only depend on the data.

(v) Estimating \tilde{Z}_{n+1} . We proceed similarly as in step (iv). Analogously to (3.12) we get an estimate for \tilde{Z}_{n+1} of the form

$$\tilde{Z}_{n+1} \leq m^{q_1^+} \sum_{i=1}^m \left[\int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^{q_{1,i}^+} \xi_i^{q_{1,i}^+} d\sigma + \int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^{q_{1,i}^-} \xi_i^{q_{1,i}^-} d\sigma \right], \tag{3.18}$$

where $q_{1,i}^- = \min_{B_i \cap \Gamma} q_1(x)$. Note that $p_i^- \leq q_{1,i}^- \leq q_{1,i}^+ < (p_i^-)^*$ for $i = 1, \dots, m$.

Let now $i \in \{1, \dots, m\}$ be fixed, and suppose that $r \in \{q_{1,i}^-, q_{1,i}^+\}$, that is we have $p_i^- \leq r < (p_i^-)^*$ and $r \leq q^+$. Define $s = s_i(r) \in (1, N)$ by means of

$$s_* = \begin{cases} \frac{r+(p_i^-)^*}{2} & \text{if } (p_i^-)^* < \infty, \\ r+1 & \text{if } (p_i^-)^* = \infty. \end{cases}$$

Then $s < p_i^- \leq r < s_* < (p_i^-)_*$. Since the trace operator maps $W^{1,s}(\Omega)$ boundedly into $L^r(\Gamma)$, we have, using Hölder's inequality,

$$\begin{aligned} & \int_{\Gamma} ((u - k_{n+1})_+ \xi_i)^r d\sigma \\ & \leq \hat{C}^r \left[\int_{\Omega} (|\nabla[(u - k_{n+1})_+ \xi_i]|^s + |(u - k_{n+1})_+ \xi_i|^s) dx \right]^{\frac{r}{s}} \\ & \leq \hat{C}^{q^+} \left[\int_{\Omega} (|\nabla[(u - k_{n+1})_+ \xi_i]|^{p_i^-} + |(u - k_{n+1})_+ \xi_i|^{p_i^-}) dx \right]^{\frac{r}{p_i^-}} |A_{k_{n+1}}| \left(1 - \frac{s}{p_i^-}\right)^{\frac{r}{s}}, \end{aligned}$$

where \hat{C} denotes the maximum of 1 and the norms of the trace maps $\gamma : W^{1,s}(\Omega) \rightarrow L^r(\Gamma)$ when r runs through set $\bigcup_{j=1}^m \{q_{1,j}^-, q_{1,j}^+\}$. The right-hand side of the last inequality can be estimated to get

$$\begin{aligned} & \int_{\Gamma} ((u - k_{n+1})_+ \xi_i)^r d\sigma \\ & \leq d_9 \left(\int_{\Omega} |\nabla(u - k_{n+1})_+|^{p_i^-} \xi_i^{p_i^-} dx \right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}| \left(1 - \frac{s}{p_i^-}\right)^{\frac{r}{s}} \\ & \quad + d_{10} \left(\int_{A_{k_{n+1}}} u^{q_0(x)} dx \right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}| \left(1 - \frac{s}{p_i^-}\right)^{\frac{r}{s}} \end{aligned} \quad (3.19)$$

with positive constants d_9, d_{10} that only depend on the data.

From (3.5), (3.11), (3.18), and (3.19) with $r = q_{1,i}^+$ and $r = q_{1,i}^-$, respectively, for $i = 1, \dots, m$, we infer that

$$\begin{aligned} \tilde{Z}_{n+1} & \leq m^{q_1^+} \sum_{i=1}^m \left[d_9 \left(d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{q_{1,i}^+}{p_i^-}} |A_{k_{n+1}}| \left(1 - \frac{s_i(q_{1,i}^+)}{p_i^-}\right)^{\frac{q_{1,i}^+}{s_i(q_{1,i}^+)}} \right. \\ & \quad + d_{10} \left(2^{q_0^+(n+2)} Z_n \right)^{\frac{q_{1,i}^+}{p_i^-}} |A_{k_{n+1}}| \left(1 - \frac{s_i(q_{1,i}^+)}{p_i^-}\right)^{\frac{q_{1,i}^+}{s_i(q_{1,i}^+)}} \\ & \quad + d_9 \left(d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{q_{1,i}^-}{p_i^-}} |A_{k_{n+1}}| \left(1 - \frac{s_i(q_{1,i}^-)}{p_i^-}\right)^{\frac{q_{1,i}^-}{s_i(q_{1,i}^-)}} \\ & \quad \left. + d_{10} \left(2^{q_0^+(n+2)} Z_n \right)^{\frac{q_{1,i}^-}{p_i^-}} |A_{k_{n+1}}| \left(1 - \frac{s_i(q_{1,i}^-)}{p_i^-}\right)^{\frac{q_{1,i}^-}{s_i(q_{1,i}^-)}} \right]. \end{aligned} \quad (3.20)$$

Put $\tilde{\eta} = \max\left(\frac{s_1(q_{1,1}^+)}{p_1^-}, \dots, \frac{s_m(q_{1,m}^+)}{p_m^-}\right)$. Similarly to (3.16) we have for $r \in \{q_{1,i}^+, q_{1,i}^-\}$

$$\begin{aligned} & \left(d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{r}{p_i^-}} \leq (d_4)^{\frac{q^+}{p^-}} \left(a^{\frac{q^+}{p^-}} \right)^n (Y_n + Y_n^{\frac{q^+}{p^-}}), \\ & \left(2^{q_0^+(n+2)} Z_n \right)^{\frac{r}{p_i^-}} \leq \left(2^{\frac{(q^+)^2(n+2)}{p^-}} \right) (Y_n + Y_n^{\frac{q^+}{p^-}}), \\ & |A_{k_{n+1}}| \left(1 - \frac{s_i(r)}{p_i^-}\right)^{\frac{r}{s_i(r)}} \leq 2^{q^+(n+1)} \left(\frac{1}{k^{q_0^-}} \right)^{1-\tilde{\eta}} (Y_n^{q^+} + Y_n^{1-\tilde{\eta}}). \end{aligned} \quad (3.21)$$

Finally, (3.20) and (3.21) imply that

$$\tilde{Z}_{n+1} \leq d_{11}d_{12} \frac{1}{k^{q_0^-(1-\tilde{\eta})}} \left(Y_n^{q^+1} + Y_n^{2-\tilde{\eta}} + Y_n^{q^+ + \frac{q^+}{p^-}} + Y_n^{1 + \frac{q^+}{p^-} - \tilde{\eta}} \right) \quad (3.22)$$

where d_{11} and d_{12} are positive constants that only depend on the data.

(vi) The iterative inequality for Y_n . Recall that $Y_n = Z_n + \tilde{Z}_n$. Hence (3.17) and (3.22) yield

$$\begin{aligned} Y_{n+1} &\leq Kb^n \frac{1}{k^{q_0^-(1-\hat{\eta})}} \left(Y_n^2 + Y_n^{2-\eta} + Y_n^{1 + \frac{q^+}{p^-}} + Y_n^{1 + \frac{q^+}{p^-} - \eta} \right. \\ &\quad \left. + Y_n^{q^+} + Y_n^{2-\tilde{\eta}} + Y_n^{q^+ + \frac{q^+}{p^-}} + Y_n^{1 + \frac{q^+}{p^-} - \tilde{\eta}} \right) \\ &\leq 8Kb^n \frac{1}{k^{q_0^-(1-\hat{\eta})}} (Y_n^{1+\delta_1} + Y_n^{1+\delta_2}) \end{aligned}$$

with $K = \max(d_7, d_{11})$, $b = \max(d_8, d_{12})$, $\hat{\eta} = \max(\eta, \tilde{\eta})$, and where $0 < \delta_1 \leq \delta_2$ are given by

$$\begin{aligned} \delta_1 &= \min \left(1, 1 - \eta, \frac{q^+}{p^-}, \frac{q^+}{p^-} - \eta, q^+, 1 - \tilde{\eta}, q^+ + \frac{q^+}{p^-} - 1, \frac{q^+}{p^-} - \tilde{\eta} \right), \\ \delta_2 &= \max \left(1, 1 - \eta, \frac{q^+}{p^-}, \frac{q^+}{p^-} - \eta, q^+, 1 - \tilde{\eta}, q^+ + \frac{q^+}{p^-} - 1, \frac{q^+}{p^-} - \tilde{\eta} \right). \end{aligned}$$

Without loss of generality we may assume that $b > 1$. Now we may apply Lemma 2.1, which says that $Y_n \rightarrow 0$ as $n \rightarrow \infty$ provided

$$Y_0 = \int_{\Omega} (u - k)_+^{q_0(x)} dx + \int_{\Gamma} (u - k)_+^{q_1(x)} d\sigma \leq \left(\frac{16K}{k^{q_0^-(1-\hat{\eta})}} \right)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}}. \quad (3.23)$$

Relation (3.23) is clearly satisfied if

$$\int_{\Omega} u_+^{q_0(x)} dx + \int_{\Gamma} u_+^{q_1(x)} d\sigma \leq \left(\frac{16K}{k^{q_0^-(1-\hat{\eta})}} \right)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}}. \quad (3.24)$$

Hence, if we choose k such that

$$k = \max \left(1, \left[(16K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1^2}} \left(\int_{\Omega} u_+^{q_0(x)} dx + \int_{\Gamma} u_+^{q_1(x)} d\sigma \right) \right]^{\frac{\delta_1}{q_0^-(1-\hat{\eta})}} \right),$$

then (3.24) and in particular (3.23) are satisfied. Since $k_n \rightarrow 2k$ as $n \rightarrow \infty$ we obtain

$$\operatorname{ess\,sup}_{\Omega} u \leq 2k = 2 \max \left(1, \left[(16K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1^2}} \left(\int_{\Omega} u_+^{q_0(x)} dx + \int_{\Gamma} u_+^{q_1(x)} d\sigma \right) \right]^{\frac{\delta_1}{q_0^-(1-\hat{\eta})}} \right).$$

Tracing back the constants, we see that the first part of the theorem is proved. The supersolution case can be done analogously, replacing u with $-u$ and A_k with \tilde{A}_k , and using Lemma 3.2 instead of Lemma 3.1. This completes the proof. \square

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