RESONANT $(p,2)$-EQUATIONS WITH CONCAVE TERMS

NIKOLAOS S. PAPAGEORGIOU AND PATRICK WINKERT

Abstract. We consider a nonlinear, nonhomogeneous parametric elliptic Dirichlet equation driven by the sum of a p-Laplacian and a Laplacian (so-called $(p,2)$-equation) and with a nonlinearity involving a concave term which enters with a negative sign. By applying variational methods along with truncation and comparison techniques as well as Morse theory (critical groups), we show that the problem under consideration has at least five nontrivial solutions (four of them have constant sign) for all sufficiently small values of the parameter.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < q < 2 < p < \infty$. We study the following nonlinear nonhomogeneous parametric Dirichlet problem

$$\begin{aligned} -\Delta_p u - \Delta u &= f(x,u) - \lambda |u|^{q-2}u \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{$(P)_\lambda$}$$

where Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(\|\nabla u\|_{\mathbb{R}^N}^{p-2} \nabla u\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Here $\lambda > 0$ is a parameter to be specified and since $1 < q < 2 < p < \infty$ the right-hand side in problem $(P)_\lambda$ contains a concave term given through $-\lambda|u|^{q-2}u$. The perturbation $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, $x \mapsto f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \mapsto f(x,s)$ is continuous for a.a. $x \in \Omega$) being $(p-1)$-linear near $\pm\infty$ and resonance can occur with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$. The aim of this work is to prove the existence of multiple solutions as the parameter $\lambda > 0$ varies.

The study of elliptic problems with concave nonlinearities started with the seminal work of Ambrosetti-Brezis-Cerami [3], who examined a semilinear equation driven by the Dirichlet Laplacian and with a parametric reaction of the special form

$$f(s) = \lambda |s|^{q-2}s + |s|^{r-2}s, \tag{1.1}$$

where

$$1 < q < 2 < r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N = 1,2 \end{cases}.$$

In (1.1) we have the competing effects of two distinct nonlinearities of different nature meaning that there is a concave term $\lambda|s|^{q-2}s$ and also a convex one

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$|s|^{r-2}s$. The authors of [3] were interested to find positive solutions and proved that the problem has two positive solutions provided $\lambda > 0$ is sufficiently small. Additional results for problems with combined nonlinearities as above were obtained by Bartsch-Willem [4], Li-Wu-Zhou [17], and Wang [24]. Extensions to equations driven by the Dirichlet p-Laplacian can be found in García Azorero-Peral Alonso-Manfredi [10], Gasiński-Papageorgiou [12], Guo-Zhang [13], Hu-Papageorgiou [14], and Marano-Papageorgiou [19]. In all of the aforementioned works, the parametric concave term enters in the reaction with a positive sign. In our problem $(P)_\lambda$ the parametric concave term appears in the reaction with a negative sign. This produces a different geometry for the problem and therefore leads to a different multiplicity theorem. Semilinear problems with such a concave term were investigated by de Paiva-Massa [9], Papageorgiou-Rădulescu [21], and Perera [22].

We mention that equations involving the sum of a p-Laplacian and a Laplacian (also known as $(p,2)$-equations) arise in mathematical physics, see, for example, the works of of Benci-D'Avenia-Fortunato-Pisani [5] (quantum physics) and Cherfils-Il′yasov [7] (plasma physics).

Our approach is variational based on critical point theory coupled with suitable truncation and comparison techniques as well as Morse theory (critical groups). In the next section, for the reader's convenience, we review the main mathematical tools that we will use in the sequel.

2. Preliminaries

Let X be a Banach space and X^* its topological dual while $\langle \cdot, \cdot \rangle$ denotes the duality brackets to the pair (X^*, X).

Definition 2.1. *The functional $\varphi \in C^1(X)$ fulfills the Palais-Smale condition (the PS-condition for short) if the following holds: Every sequence $(u_n)_{n\geq 1} \subseteq X$ such that $(\varphi(u_n))_{n\geq 1}$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.*

This compactness-type condition on φ leads to a deformation theorem which is the main ingredient in the minimax theory of the critical values of φ. A basic result in that theory is the so-called mountain-pass theorem.

Theorem 2.2. *Let $\varphi \in C^1(X)$ be a functional satisfying the PS-condition and let $u_1, u_2 \in X, \|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$. Then $c \geq m_\rho$ with c being a critical value of φ.

In the analysis of problem $(P)_\lambda$ in addition to the Sobolev space $W_0^{1,p}(\Omega)$ we will also use the ordered Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}$$

and its positive cone

$$C_0^1(\overline{\Omega})_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) = \left\{u \in C_0^1(\overline{\Omega}) : u(x) > 0 \ \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial n}(x) < 0 \ \forall x \in \partial\Omega\right\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$.

Throughout this paper we denote the norm of $W_0^{1,p}(\Omega)$ by $\|\cdot\|_{W_0^{1,p}(\Omega)}$ and thanks to the Poincaré inequality it holds $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$ for all $u \in W_0^{1,p}(\Omega)$, where $\|\cdot\|_p$ stands for the usual L^p-norm. The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product of $\mathbb{R}^N$.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying a subcritical growth with respect to the second argument, that is

$$|f_0(x,s)| \leq a(x)\left(1+|s|^{r-1}\right) \qquad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, and $1 < r < p^*$, where p^* is the critical exponent of p given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

Let $F_0(x,s) = \int_0^s f_0(x,t)dt$ and let $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_0(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F_0(x,u)dx.$$

The next result is a special case of a more general theorem of Aizicovici-Papageorgiou-Staicu [2] and essentially is an outgrowth of the nonlinear regularity theory (see Ladyzhenskaya-Ural′tseva [15], Lieberman [16]).

Theorem 2.3. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0+h) \quad \text{for all } h \in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0+h) \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \|h\|_{W_0^{1,p}(\Omega)} \leq \rho_1.$$

Remark 2.4. *The first result in this direction was obtained by Brezis-Nirenberg [6]. Subsequently important extensions were proved by García Azorero-Peral Alonso-Manfredi [10], Guo-Zhang [13], and Winkert [25].*

Given $1 < r < \infty$, we denote by $\Delta_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ with $\frac{1}{r}+\frac{1}{r'} = 1$ the r-Laplacian defined by

$$\langle \Delta_r u, v\rangle = \int_\Omega \|\nabla u\|_{\mathbb{R}^N}^{r-2}(\nabla u, \nabla v)_{\mathbb{R}^N}\,dx \quad \text{for all } u,v \in W_0^{1,r}(\Omega). \tag{2.1}$$

If $r = 2$, then $\Delta_r = \Delta$ becomes the well-known Laplace operator and we have $\Delta \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$, where $\mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ denotes the vector space of all bounded linear operators from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. The next proposition summarizes the main properties of the map $-\Delta_r$ (see Gasiński-Papageorgiou [11]).

Proposition 2.5. *If $\Delta_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ with $1 < r < \infty$, $\frac{1}{r}+\frac{1}{r'} = 1$, is defined by (2.1), then Δ_r is bounded (in the sense that it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone) and of type $(S)_+$, i.e., if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle -\Delta_r u_n, u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

Let $\hat{\lambda}_1(p)$ be the first eigenvalue of the negative Dirichlet p-Laplacian $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$ which has the subsequent properties:

- $\hat{\lambda}_1(p)$ is positive, simple and isolated;
- $$\hat{\lambda}_1(p) = \inf\left[\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right]. \tag{2.2}$$

The infimum in (2.2) is realized on the one dimensional eigenspace whose elements do not change sign which easily follows from the representation in (2.2). Denote by $\hat{u}_1(p)$ the L^p-normalized eigenfunction (i.e. $\|\hat{u}_1(p)\|_p = 1$) associated to $\hat{\lambda}_1(p)$, the nonlinear regularity theory implies that $\hat{u}_1(p) \in C_0^1(\overline{\Omega})$ and the usage of the nonlinear maximum principle (see Gasiński-Papageorgiou [11, pp. 737–738]) yields $\hat{u}_1(p) \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

In addition to $\hat{\lambda}_1(p) > 0$, the Lusternik-Schnirelmann minimax scheme gives a whole strictly increasing sequence $\left(\hat{\lambda}_k(p)\right)_{k\geq 1}$ of eigenvalues of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$ such that $\hat{\lambda}_k(p) \to +\infty$ as $k \to \infty$. If $p \neq 2$ we do not know if this sequence exhausts the whole spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$ but in case $N = 1$ (ordinary differential equations) or $p = 2$ (linear eigenvalue problem) the answer is positive. In the case $p = 2$ we denote by $E\left(\hat{\lambda}_k(2)\right), k \geq 1$, the finite dimensional eigenspace corresponding to the eigenvalue $\hat{\lambda}_k(2)$. Applying classical regularity theory we have that $E\left(\hat{\lambda}_k(2)\right) \subseteq C_0^1(\overline{\Omega})$ for all $k \geq 1$ and the eigenspace has the so-called unique continuation property (ucp for short) meaning that if $u \in E\left(\hat{\lambda}_k(2)\right)$ vanishes on a set of positive Lebesgue measure, then $u(x) = 0$ for all $x \in \overline{\Omega}$. For every $k \geq 1$ we set

$$\overline{H}_k = \bigoplus_{i=1}^{k} E\left(\hat{\lambda}_i(2)\right) \quad \text{and} \quad \hat{H}_k = \overline{H}_k^{\perp} = \overline{\bigoplus_{i\geq k+1} E\left(\hat{\lambda}_i(2)\right)}.$$

In the linear case we have a variational characterization for all eigenvalues, namely

$$\hat{\lambda}_1(2) = \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0\right] \tag{2.3}$$

and for $k \geq 2$

$$\begin{aligned}\hat{\lambda}_k(2) &= \max\left[\frac{\|\nabla \overline{u}\|_2^2}{\|\overline{u}\|_2^2} : \overline{u} \in \overline{H}_k, \overline{u} \neq 0\right]\\ &= \min\left[\frac{\|\nabla \hat{u}\|_2^2}{\|\hat{u}\|_2^2} : \hat{u} \in \hat{H}_{k-1}, \hat{u} \neq 0\right].\end{aligned} \tag{2.4}$$

Taking into account the ucp of the eigenspaces along with (2.3), (2.4) we obtain the subsequent lemma.

Lemma 2.6.

(a) If $k \geq 1$, $\vartheta \in L^\infty(\Omega)_+, \vartheta(x) \leq \hat{\lambda}_k(2)$ a.e. in Ω with $\vartheta \neq \hat{\lambda}_k(2)$, then there exists $\hat{\xi}_0 > 0$ such that

$$\|\nabla \hat{u}\|_2^2 - \int_\Omega \vartheta \hat{u}^2 dx \geq \hat{\xi}_0 \|\hat{u}\|_{H_0^1(\Omega)}^2 \quad \textit{for all } \hat{u} \in \hat{H}_{k-1}.$$

(b) If $k \geq 1$, $\vartheta \in L^\infty(\Omega)_+, \vartheta(x) \geq \hat{\lambda}_k(2)$ a.e. in Ω with $\vartheta \neq \hat{\lambda}_k(2)$, then there exists $\hat{\xi}_1 > 0$ such that

$$\|\nabla \overline{u}\|_2^2 - \int_\Omega \vartheta \overline{u}^2 dx \leq -\hat{\xi}_1 \|\overline{u}\|^2_{H_0^1(\Omega)} \quad \text{for all } \overline{u} \in \overline{H}_k.$$

Using the properties of $\hat{\lambda}_1(p)$ we derive the following result (see, for example, Papageorgiou-Kyritsi [20, p. 356]).

Lemma 2.7. *Let $\vartheta \in L^\infty(\Omega)_+$ be such that $\vartheta(x) \leq \hat{\lambda}_1(p)$ a.e. in Ω and $\vartheta \neq \hat{\lambda}_1(p)$. Then there exists a number $\hat{\xi}_2 > 0$ such that*

$$\|\nabla u\|_p^p - \int_\Omega \vartheta |u|^p dx \geq \hat{\xi}_2 \|u\|^p_{W_0^{1,p}(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Next, we briefly recall some basic definitions and facts about Morse theory related to critical points. To this end, let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets.

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \leq c\} && \text{(the sublevel set of } \varphi \text{ at } c\text{)},\\ K_\varphi &= \{u \in X : \varphi'(u) = 0\} && \text{(the critical set of } \varphi\text{)},\\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\} && \text{(the critical set of } \varphi \text{ at the level } c\text{)}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$ we denote by $H_k(Y_1, Y_2)$ the $k \stackrel{th}{=}$-relative singular homology group of the pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated $u_0 \in K_\varphi^c$ are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}) \quad \text{for all integers } k \geq 0,$$

where U is a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology theory implies that the definition of critical groups above is independent of the particular choice of the neighborhood U.

If $u_0 \in X$ is a local minimizer of φ, then

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0, \tag{2.5}$$

where $\delta_{k,0}$ is the Kronecker symbol, that is

$$\delta_{k,0} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}.$$

For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is well known that

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

The Lebesgue measure on $\mathbb{R}^N$ will be denoted by $|\cdot|_N$. Finally, for any Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ we define the Nemytskij operator $N_h : L^p(\Omega) \to (L^p(\Omega))^*$ corresponding to the function h by

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)).$$

3. Constant sign solutions

In this section we prove the existence of constant sign solutions for problem $(P)_\lambda$. We impose the following conditions on the perturbation $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x,0) = 0$ for a.a. $x \in \Omega$ and

(i) $|f(x,s)| \leq a(x)\left(1+|s|^{p-1}\right)$ for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \hat{\lambda}_1(p)$ uniformly for a.a. $x \in \Omega$ and there exists $\xi_0 > 0$ such that

$$f(x,s)s - pF(x,s) \geq -\xi_0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R},$$

where $F(x,s) = \int_0^s f(x,t)dt$;

(iii) there exist functions $\eta, \hat{\eta} \in L^\infty(\Omega)_+$ such that

$$\hat{\lambda}_1(2) \leq \eta(x) \quad \text{a.e. in } \Omega, \eta \neq \hat{\lambda}_1(2)$$

and

$$\eta(x) \leq \liminf_{s\to 0} \frac{f(x,s)}{s} \leq \limsup_{s\to 0} \frac{f(x,s)}{s} \leq \hat{\eta}(x)$$

uniformly for a.a. $x \in \Omega$;

(iv) $f(x,\cdot)$ is locally lower Lipschitz for a.a. $x \in \Omega$, that is, for every compact set $K \subseteq \mathbb{R}$, there exists a constant $c_K > 0$ such that

$$f(x,s_1) - f(x,s_2) \geq -c_K|s_1 - s_2| \quad \text{for all } s_1, s_2 \in K.$$

Remark 3.1. *Hypothesis H_1(ii) implies that we can have resonance asymptotically at $\pm\infty$ with respect to $\hat{\lambda}_1(p) > 0$.*

In order to prove the existence of constant sign solutions we consider the positive and negative truncations of the reaction in problem $(P)_\lambda$ for $\lambda > 0$, namely the Carathéodory functions

$$g_\lambda^\pm(x,s) = f(x, \pm s^\pm) \mp \lambda\left(s^\pm\right)^{q-1}.$$

We set $G_\lambda^\pm(x,s) = \int_0^s g_\lambda^\pm(x,t)dt$ and consider the C^1-functionals $\varphi_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega G_\lambda^\pm(x,u)dx.$$

The corresponding energy functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ to problem $(P)_\lambda$ is defined by

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(x,u)dx,$$

which is of class C^1 as well. First, we will see that the functionals stated above are coercive.

Proposition 3.2. *Let hypotheses H_1 be satisfied and let $\lambda > 0$. Then the functionals $\varphi_\lambda^\pm$ and φ_λ are coercive.*

Proof. We will show the proof only for φ_λ^+, the proofs for the other functionals work similarly. Arguing by contradiction we suppose that φ_λ^+ is not coercive. Then we find a sequence $(u_n)_{n\geq 1}\subseteq W_0^{1,p}(\Omega)$ and a number $M_1>0$ such that

$$\|u_n\|_{W_0^{1,p}(\Omega)}\to\infty \quad\text{and}\quad \varphi_\lambda^+(u_n)\leq M_1.$$

The second relation gives

$$\frac{1}{p}\|\nabla u_n\|_p^p+\frac{1}{2}\|\nabla u_n\|_2^2-\int_\Omega G_\lambda^+(x,u_n)dx\leq M_1 \quad\text{for all } n\geq 1. \tag{3.1}$$

Taking $y_n=\frac{u_n}{\|u_n\|_{W_0^{1,p}(\Omega)}}$ implies $\|y_n\|_{W_0^{1,p}(\Omega)}=1$ and we may assume that

$$y_n\rightharpoonup y \quad\text{in } W_0^{1,p}(\Omega) \quad\text{and}\quad y_n\to y \quad\text{in } L^p(\Omega) \tag{3.2}$$

with some $y\in W_0^{1,p}(\Omega)$. Applying the representation of y_n inequality (3.1) becomes

$$\frac{1}{p}\|\nabla y_n\|_p^p-\int_\Omega\frac{G_\lambda^+(x,u_n)}{\|u_n\|^p_{W_0^{1,p}(\Omega)}}dx\leq\frac{M_1}{\|u_n\|^p_{W_0^{1,p}(\Omega)}} \quad\text{for all } n\geq 1. \tag{3.3}$$

Because of hypothesis H_1(i) we have that

$$\left(\frac{G_\lambda^+(\cdot,u_n(\cdot))}{\|u_n\|^p_{W_0^{1,p}(\Omega)}}\right)_{n\geq 1}\subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

Taking into account the Dunford-Pettis theorem along with assumption H_1(ii) we obtain

$$\frac{G_\lambda^+(\cdot,u_n(\cdot))}{\|u_n\|^p_{W_0^{1,p}(\Omega)}}\rightharpoonup\frac{1}{p}\vartheta\left(y^+\right)^p \quad\text{in } L^2(\Omega) \tag{3.4}$$

with $\vartheta\in L^\infty(\Omega)$ satisfying $\vartheta(x)\leq\hat{\lambda}_1(p)$ a.e. in Ω. Passing to the limit in (3.3) as $n\to\infty$ and applying (3.2) as well as (3.4) yields

$$\|\nabla y\|_p^p\leq\int_\Omega\vartheta\left(y^+\right)^p dx, \tag{3.5}$$

which implies

$$\left\|\nabla y^+\right\|_p^p\leq\int_\Omega\vartheta\left(y^+\right)^p dx. \tag{3.6}$$

Suppose now that $\vartheta\neq\hat{\lambda}_1(p)$. Then from (3.6) and Lemma 2.7 we get $y^+=0$. So inequality (3.5) implies $y^-=0$, that is $y=0$. Then, using (3.3), we see that

$$y_n\to 0 \quad\text{in } W_0^{1,p}(\Omega),$$

a contradiction to the fact that $\|y_n\|_{W_0^{1,p}(\Omega)}=1$ for all $n\geq 1$.

Now we assume that $\vartheta(x)=\hat{\lambda}_1(p)$ a.e. in Ω. Then (3.6) and (2.2) give

$$\left\|\nabla y^+\right\|_p^p=\hat{\lambda}_1(p)\left\|y^+\right\|_p^p$$

which means that

$$y^+=\xi\hat{u}_1(p) \quad\text{for some } \xi\geq 0.$$

If $\xi=0$, then $y^+=0$ and due to (3.5) $y=0$. Hence, because of (3.3), $y_n\to 0$ in $W_0^{1,p}(\Omega)$ which is a contradiction since $\|y_n\|_{W_0^{1,p}(\Omega)}=1$ for all $n\geq 1$.

If $\xi > 0$, then $y^+ \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so $y^+(x) > 0$ for all $x \in \Omega$. Since y^+ is the limit of y_n^+ in $W_0^{1,p}(\Omega)$ (see (3.2)) and $y_n^+ = \frac{u_n^+}{\|u_n\|_{W_0^{1,p}(\Omega)}}$ it follows that

$$u_n^+(x) \to +\infty \quad \text{for a.a. } x \in \Omega. \tag{3.7}$$

Thanks to hypothesis H_1(ii) and since $q < p$ we further obtain for a.a. $x \in \Omega$ and for all $u > 0$

$$\begin{aligned}
\frac{d}{du}\frac{G_\lambda^+(x,u)}{u^p} &= \frac{f(x,u)u^p - \lambda u^{q+p-1} - pF(x,u)u^{p-1} + \frac{\lambda p}{q}u^{q+p-1}}{u^{2p}}\\
&= \frac{f(x,u)u - \lambda u^q - pF(x,u) + \frac{\lambda p}{q}u^q}{u^{p+1}}\\
&\geq -\frac{\xi_0}{u^{p+1}}.
\end{aligned}$$

We conclude

$$\frac{G_\lambda^+(x,y)}{y^p} - \frac{G_\lambda^+(x,u)}{u^p} \geq \frac{\xi_0}{p}\left[\frac{1}{y^p} - \frac{1}{u^p}\right] \tag{3.8}$$

for a.a. $x \in \Omega$ and for all $y \geq u > 0$. From hypothesis H_1(ii) we see at once that

$$\limsup_{s\to\pm\infty}\frac{pF(x,s)}{|s|^p} \leq \hat{\lambda}_1(p) \quad \text{uniformly for a.a. } x \in \Omega.$$

Then, passing in (3.8) to the limit as $y \to +\infty$, since $q < p$, we derive

$$\frac{\hat{\lambda}_1(p)}{p} - \frac{G_\lambda^+(x,u)}{u^p} \geq -\frac{\xi_0}{p}\frac{1}{u^p} \quad \text{for a.a. } x \in \Omega \text{ and for all } u > 0,$$

which implies

$$pF(x,u) - \frac{\lambda p}{q}u^q - \hat{\lambda}_1(p)u^p \leq \xi_0 \quad \text{for a.a. } x \in \Omega \text{ and for all } u \geq 0. \tag{3.9}$$

Inequality (3.1) can be written as

$$\frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2}\|\nabla u_n\|_2^2 \leq M_1 + \int_\Omega G_\lambda^+(x,u_n)dx \quad \text{for all } n \geq 1,$$

which, due to (2.2) and (3.9), gives

$$\begin{aligned}
\frac{p}{2}\hat{\lambda}_1(2)\left\|u_n^+\right\|_2^2 &\leq M_1 p + \int_\Omega\left[pF\left(x,u_n^+\right) - \frac{\lambda p}{q}\left(u_n^+\right)^q - \hat{\lambda}_1(p)\left(u_n^+\right)^p\right]dx\\
&\leq M_2
\end{aligned}$$

with $M_2 = M_1 p + \xi_0|\Omega|_N > 0$ and for all $n \geq 1$. This implies

$$\int_\Omega \left(u_n^+\right)^2 dx \leq \frac{2M_2}{p\hat{\lambda}_1(2)} \quad \text{for all } n \geq 1. \tag{3.10}$$

On the other side, from (3.7) and Fatou's Lemma, we have

$$\int_\Omega \left(u_n^+\right)^2 dx \to +\infty \quad \text{as } n \to \infty. \tag{3.11}$$

Comparing (3.10) and (3.11) we reach a contradiction. This proves that φ_λ^+ is coercive. $\square$

In general coercivity does not imply the PS-condition (see, for example, Gasiński-Papageorgiou [11, Example 5.1.15]). However, for the functionals $\varphi_\lambda^\pm$ and φ_λ this implication is true as stated in the next proposition, which is a consequence of Proposition 2.2 of Marano-Papageorgiou [18]. For completeness we provide the proof.

Proposition 3.3. *If $\varphi_\lambda^\pm$ and φ_λ are coercive, then they satisfy the PS-condition.*

Proof. The proof will be given only for φ_λ^+, the other ones work similarly. Suppose $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is a PS-sequence, that is

$$\left|\varphi_\lambda^+(u_n)\right| \leq M_4 \quad \text{for some } M_4>0, \text{ for all } n\geq 1, \tag{3.12}$$

$$\left(\varphi_\lambda^+\right)'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n\to\infty. \tag{3.13}$$

The assertion in (3.12) along with the coercivity of φ_λ^+ implies that $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Therefore, we may assume that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega). \tag{3.14}$$

From (3.13) it follows

$$\left|\langle -\Delta_p u_n, h\rangle + \langle -\Delta u_n, h\rangle - \int_\Omega g_\lambda^+(x,u_n) h dx\right| \leq \varepsilon_n \|h\|_{W_0^{1,p}(\Omega)},$$

for all $h\in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. Now, choosing $h=u_n-u\in W_0^{1,p}(\Omega)$, passing to the limit as $n\to\infty$, and using the convergence properties in (3.14) we obtain

$$\lim_{n\to\infty}\left[\langle -\Delta_p u_n, u_n-u\rangle + \langle -\Delta u_n, u_n-u\rangle\right] = 0,$$

which by the monotonicity of $-\Delta$ implies that

$$\limsup_{n\to\infty}\left[\langle -\Delta_p u_n, u_n-u\rangle + \langle -\Delta u, u_n-u\rangle\right] \leq 0.$$

Applying again (3.14) we infer from the last relation

$$\limsup_{n\to\infty}\langle -\Delta_p u_n, u_n-u\rangle \leq 0,$$

which by the $(\mathrm{S})_+$-property of $-\Delta_p$ (see Proposition 2.5) results in $u_n\to u$ in $W_0^{1,p}(\Omega)$. Hence, φ_λ^+ fulfills the PS-condition. $\square$

Proposition 3.4. *If hypotheses H_1 hold, then we can find $\lambda^*>0$ such that for all $\lambda\in(0,\lambda^*)$ there exists $t^*=t^*(\lambda)$ for which*

$$\varphi_\lambda\left(\pm t^*\hat{u}_1(2)\right) < 0.$$

Proof. Given $\varepsilon>0$, by virtue of hypotheses H_1(i), (iii), there exists $c_1=c_1(\varepsilon)>0$ such that

$$F(x,s) \geq \frac{1}{2}\left(\eta(x)-\varepsilon\right)s^2 - c_1|s|^p \quad \text{for a.a. } x\in\Omega \text{ and for all } s\in\mathbb{R}. \tag{3.15}$$

By means of (3.15) we have for $t > 0$

$$
\begin{aligned}
\varphi_\lambda\left(t\hat{u}_1(2)\right) &= \frac{t^p}{p}\|\nabla \hat{u}_1(2)\|_p^p + \frac{t^2}{2}\|\nabla \hat{u}_1(2)\|_2^2 + \frac{\lambda t^q}{q}\|\hat{u}_1(2)\|_q^q \\
&\quad - \int_\Omega F\left(x, t\hat{u}_1(2)\right) dx \\
&\leq \frac{t^2}{2}\left[\int_\Omega \left(\hat{\lambda}_1(2) - \eta(x)\right)\left(\hat{u}_1(2)\right)^2 dx + \varepsilon\right] + c_2\left[t^p + \lambda t^q\right]
\end{aligned}
$$

for some $c_2 > 0$. Thanks to hypothesis H_1(iii) and since $\hat{u}_1(2) \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ we conclude

$$
\xi_* = \int_\Omega \left(\eta(x) - \hat{\lambda}_1(2)\right)\left(\hat{u}_1(2)\right)^2 dx > 0.
$$

Choosing $\varepsilon \in (0, \xi_*)$ we have

$$
\begin{aligned}
\varphi_\lambda\left(t\hat{u}_1(2)\right) &\leq -c_3 t^2 + c_2\left[t^p + \lambda t^q\right] \\
&= \left[c_2\left(t^{p-2} + \lambda t^{q-2}\right) - c_3\right] t^2
\end{aligned}
\tag{3.16}
$$

for some $c_3 > 0$ and for all $t > 0$.

Let $\beta_\lambda(t) = t^{p-2} + \lambda t^{q-2}$ for all $t > 0$. Obviously, $\beta_\lambda \in C^1(0,\infty)$ and since $q < 2 < p$ it follows

$$
\beta_\lambda(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.
$$

Hence, we find a number $t_0 \in (0, +\infty)$ such that

$$
\beta\left(t_0\right) = \inf\left[\beta_\lambda(t) : t > 0\right] > 0.
$$

Moreover, it holds

$$
\beta_\lambda'(t_0) = \left[(p-2)t_0^{p-3} + \lambda(q-2)t_0^{q-3}\right] = 0,
$$

which implies

$$
t_0 = t_0(\lambda) = \left[\frac{\lambda(2-q)}{p-2}\right]^{\frac{1}{p-q}}.
$$

We see that $\beta_\lambda(t_0(\lambda)) \to 0$ as $\lambda \to 0^+$. Therefore, there exists a number $\lambda^* > 0$ such that

$$
\beta_\lambda\left(t_0\right) < \frac{c_3}{c_2} \quad \text{for all } \lambda \in \left(0, \lambda^*\right).
$$

Taking $t^* = t^*(\lambda) = t_0(\lambda)$, inequality (3.16) gives

$$
\varphi_\lambda\left(\pm t^* \hat{u}_1(2)\right) < 0.
$$

□

The next proposition will be helpful in verifying the mountain-pass geometry of the functionals $\varphi_\lambda^\pm$ and φ_λ.

Proposition 3.5. *Let hypotheses H_1 be satisfied and let $\lambda > 0$. Then $u = 0$ is a local minimizer of the functionals $\varphi_\lambda^\pm$ and φ_λ.*

Proof. As before we will do the proof only for φ_λ^+. By virtue of hypothesis H_1(iii) there exist numbers $c_4>0$ and $\delta>0$ such that

$$F(x,s)\le c_4 s^2 \quad \text{for a.a. } x\in\Omega \text{ and for all } s\in[0,\delta]. \tag{3.17}$$

Let $u\in C_0^1(\overline{\Omega})$ satisfy $\|u\|_{C_0^1(\overline{\Omega})}\le\delta$. Applying (3.17) it follows

$$\begin{aligned}\varphi_\lambda^+(u)&=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega G_\lambda^+(x,u)dx\\&\ge\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{2}\|\nabla u\|_2^2+\left[\frac{\lambda}{q}-c_4\|u\|_{C(\overline{\Omega})}^{2-q}\right]\left\|u^+\right\|_q^q\\&\ge\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{2}\|\nabla u\|_2^2+\left[\frac{\lambda}{q}-c_4\delta^{2-q}\right]\left\|u^+\right\|_q^q.\end{aligned} \tag{3.18}$$

Choosing $\delta>0$ such that $\delta<\left(\frac{\lambda}{qc_4}\right)^{\frac{1}{2-q}}$ we infer from (3.18)

$$\varphi_\lambda^+(u)\ge 0=\varphi_\lambda^+(0)\quad\text{for all } u\in C_0^1(\overline{\Omega}) \text{ with } \|u\|_{C_0^1(\overline{\Omega})}\le\delta.$$

This means that $u=0$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_λ^+ and because of Theorem 2.3 $u=0$ is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_λ^+. The proofs for φ_λ^- and φ_λ can be done in the same way. □

Now, we will apply the mountain-pass theorem (Theorem 2.2) and the direct method to prove the existence of at least four nontrivial constant sign solutions of $(P)_\lambda$ for all $\lambda>0$ sufficiently small whereby two of the solutions have positive sign and the other ones have negative sign. In what follows $\lambda^*>0$ denotes the number obtained in Proposition 3.4.

Proposition 3.6. *Let hypotheses H_1 be satisfied and let $\lambda\in(0,\lambda^*)$. Then problem $(P)_\lambda$ admits at least four nontrivial solutions of constant sign, namely*

$$u_0,\hat u\in C_0^1(\overline{\Omega})_+\setminus\{0\}\quad\textit{and}\quad v_0,\hat v\in-\left(C_0^1(\overline{\Omega})_+\right)\setminus\{0\}$$

such that

$$u_0(x),\hat u(x)>0\quad\textit{for all } x\in\Omega\quad\textit{and}\quad v_0(x),\hat v(x)<0\quad\textit{for all } x\in\Omega.$$

Moreover, u_0 and v_0 are local minimizers of φ_λ.

Proof. Taking into account Proposition 3.2 we know that φ_λ^+ is coercive for all $\lambda>0$. Moreover, by applying the Sobolev embedding theorem we easily verify that φ_λ^+ is sequentially weakly lower semicontinuous as well. Therefore, by virtue of the Weierstrass theorem, there exists an element $u_0\in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda^+(u_0)=\inf\left[\varphi_\lambda^+(u):u\in W_0^{1,p}(\Omega)\right]. \tag{3.19}$$

From Proposition 3.4 it follows that if $\lambda\in(0,\lambda^*)$ we can find a number $t^*=t^*(\lambda)>0$ such that

$$\varphi_\lambda\left(t^*\hat u_1(2)\right)<0,$$

which ensures, due to $\varphi_\lambda\big|_{C_0^1(\overline{\Omega})_+}=\varphi_\lambda^+\big|_{C_0^1(\overline{\Omega})_+}$ and $\hat u_1(2)\in\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, that

$$\varphi_\lambda^+\left(t^*\hat u_1(2)\right)<0.$$

Hence, because of (3.19), we obtain

$$\varphi_\lambda^+(u_0) < 0 = \varphi_\lambda^+(0),$$

implying $u_0 \neq 0$. Since u_0 is a critical point of φ_λ^+ we have

$$\left(\varphi_\lambda^+\right)'(u_0) = 0,$$

that is,

$$\langle -\Delta_p u_0, h\rangle + \langle -\Delta u_0, h\rangle = \left\langle N_{g_\lambda^+}(u_0), h\right\rangle \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.20}$$

Taking $h = -u_0^- \in W_0^{1,p}(\Omega)$ as test function in (3.20) gives $u_0 \geq 0$. Therefore, (3.20) becomes

$$\langle -\Delta_p u_0, h\rangle + \langle -\Delta u_0, h\rangle = \left\langle N_f(u_0) - \lambda u_0^{q-1}, h\right\rangle \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

meaning that u_0 solves our original problem

$$\begin{aligned} -\Delta_p u_0 - \Delta u_0 &= f(x,u_0) - \lambda u_0^{q-1} && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.21}$$

Note that $u_0 \in L^\infty(\Omega)$ (see Ladyzhenskaya-Ural'tseva [15, p. 286]) and by means of the regularity results of Lieberman [16, Theorem 1] we infer that $u_0 \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$.

Now, let $a : \mathbb{R}^N \to \mathbb{R}^N$ be the map defined by $a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\xi + \xi$. Since $p > 2$ it is easy to see that $a \in C^1\left(\mathbb{R}^N, \mathbb{R}^N\right)$. There holds

$$\nabla a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\left[I + (p-2)\frac{\xi \otimes \xi}{\|\xi\|_{\mathbb{R}^N}^2}\right] + I \quad \text{for all } \xi \in \mathbb{R}^N$$

and

$$(\nabla a(\xi) y, y)_{\mathbb{R}^N} \geq \|\xi\|_{\mathbb{R}^N}^2 \quad \text{for all } \xi, y \in \mathbb{R}^N.$$

Thanks to hypothesis H_1(iv) we may apply the tangency principle of Pucci-Serrin [23, p. 35] which gives

$$u_0(x) > 0 \quad \text{for all } x \in \Omega.$$

Claim: u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ_λ.

Arguing by contradiction suppose we can find a sequence $(u_n)_{n\geq 1} \subseteq C_0^1(\overline{\Omega})$ such that

$$u_n \to u_0 \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{and} \quad \varphi_\lambda(u_n) < \varphi_\lambda(u_0).$$

Since $\varphi_\lambda\big|_{C_0^1(\overline{\Omega})_+} = \varphi_\lambda^+\big|_{C_0^1(\overline{\Omega})_+}$ and because of (3.19) it follows

$$\begin{aligned} 0 &> \varphi_\lambda(u_n) - \varphi_\lambda(u_0)\\ &= \varphi_\lambda(u_n) - \varphi_\lambda^+(u_0)\\ &\geq \varphi_\lambda(u_n) - \varphi_\lambda^+(u_n)\\ &= \frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{\lambda}{q}\|u_n\|_q^q - \int_\Omega F(x,u_n)dx\\ &\quad - \frac{1}{p}\|\nabla u_n\|_p^p - \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\lambda}{q}\left\|u_n^+\right\|_q^q + \int_\Omega F\left(x,u_n^+\right)dx\\ &= \frac{\lambda}{q}\left\|u_n^-\right\|_q^q - \int_\Omega F\left(x,-u_n^-\right)dx. \end{aligned} \tag{3.22}$$

By virtue of hypotheses H_1(i)–(iii) there exist numbers $c_5, c_6 > 0$ such that

$$F(x,s) \leq c_5 s^2 + c_6|s|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{3.23}$$

Applying (3.23) in (3.22) yields

$$\begin{aligned} 0 &> \varphi_\lambda(u_n) - \varphi_\lambda(u_0) \\ &\geq \frac{\lambda}{q}\left\|u_n^-\right\|_q^q - c_5\left\|u_n^-\right\|_2^2 - c_6\left\|u_n^-\right\|_p^p \\ &\geq \frac{\lambda}{q}\left\|u_n^-\right\|_q^q - \left[c_5\left\|u_n^-\right\|_{C(\overline{\Omega})}^{2-q} + c_6\left\|u_n^-\right\|_{C(\overline{\Omega})}^{p-q}\right]\left\|u_n^-\right\|_q^q. \end{aligned} \tag{3.24}$$

Since $u_0 > 0$ we note that $u_n^- \to 0$ in $C(\overline{\Omega})$. Therefore, (3.24) implies the existence of a number $n_0 \geq 1$ such that

$$0 > \varphi_\lambda(u_n) - \varphi_\lambda(u_0) \geq 0 \quad \text{for all } n \geq n_0,$$

which is a contradiction. This proves the Claim.

Taking into account the Claim and Theorem 2.3 we obtain that u_0 is a $W_0^{1,p}(\Omega)$-minimizer of φ_λ^+.

From Proposition 3.5 we know that $u = 0$ is a local minimizer of φ_λ^+. We may assume that it is an isolated critical point of φ_λ^+ or otherwise we have a whole sequence of distinct positive solutions of $(P)_\lambda$. Then, from Aizicovici-Papageorgiou-Staicu [1, Proof of Proposition 29] (see also de Figueiredo [8, Theorem 5.10, p. 42]) we can find a number $\rho \in \left(0, \|u_0\|_{W_0^{1,p}(\Omega)}\right)$ sufficiently small such that

$$\varphi_\lambda^+(u_0) < 0 = \varphi_\lambda^+(0) < \inf\left[\varphi_\lambda^+(u) : \|u\|_{W_0^{1,p}(\Omega)} = \rho\right] = m_\rho^+. \tag{3.25}$$

Recall that φ_λ^+ is coercive (see Proposition 3.2). So, Proposition 3.3 implies that φ_λ^+ fulfills the PS-condition. This fact along with (3.25) permit the usage of the mountain-pass theorem stated in Theorem 2.2 to obtain an element $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\varphi_\lambda^+} \quad \text{and} \quad m_\rho^+ \leq \varphi_\lambda^+(\hat{u}). \tag{3.26}$$

Since $\hat{u} \in K_{\varphi_\lambda^+}$ we have $\left(\varphi_\lambda^+\right)'(\hat{u}) = 0$, that is

$$\langle -\Delta_p \hat{u}, h\rangle + \langle -\Delta \hat{u}, h\rangle = \left\langle N_{g_\lambda^+}(\hat{u}), h\right\rangle \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.27}$$

Taking $h = -\hat{u}^- \in W_0^{1,p}(\Omega)$ in (3.27) gives $\|\nabla \hat{u}^-\|_p^p + \|\nabla \hat{u}^-\|_2^2 = 0$. Thus, $\hat{u} \geq 0$. From (3.25) and (3.26) it follows that $\hat{u} \notin \{0, u_0\}$ and $\hat{u}$ is a positive solution of $(P)_\lambda$ with $\lambda \in (0, \lambda^*)$. As before the nonlinear regularity theory and the tangency principle imply that $\hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$ with $\hat{u}(x) > 0$ for all $x \in \Omega$.

Similarly, working with φ_λ^- instead of φ_λ^+, we show the existence of two negative constant sign solutions $v_0, \hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}$ with $v_0(x), \hat{v}(x) < 0$ for all $x \in \Omega$. $\square$

4. Five nontrivial solutions

In this section we have to strengthen the hypotheses of the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ in order to prove the existence of a fifth nontrivial solution of problem $(P)_\lambda$ for all $\lambda > 0$ sufficiently small. We suppose the following conditions.

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(x,0) = 0$ for a.a. $x \in \Omega$, $f(x,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_s(x,s)| \leq a(x)\left(1 + |s|^{p-2}\right)$ for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{s \to \pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \hat{\lambda}_1(p)$ uniformly for a.a. $x \in \Omega$ and there exists $\xi_0 > 0$ such that

$$f(x,s)s - pF(x,s) \geq -\xi_0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R},$$

where $F(x,s) = \int_0^s f(x,t)dt$;

(iii) there exist an integer $m \geq 3$ such that

$$f'_s(x,0) \in \left[\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)\right] \quad \text{a.e. in } \Omega,$$

with $f'_s(\cdot,0) \neq \hat{\lambda}_m(2)$, $f'_s(\cdot,0) \neq \hat{\lambda}_{m+1}(2)$ and

$$f'_s(x,0) = \lim_{s \to 0} \frac{f(x,s)}{s} \quad \text{uniformly for a.a. } x \in \Omega;$$

(iv) $|F(x,s)| \leq \frac{\hat{\lambda}_{m+1}(2)}{2} s^2 + \frac{\hat{\lambda}_1(p)}{p}|s|^p$ for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

Remark 4.1. *The differentiability of $f(x,\cdot)$ along with hypothesis H_2(i) imply that $f(x,\cdot)$ is locally Lipschitz.*

Let

$$V_m = \bigoplus_{i=1}^{m} E\left(\hat{\lambda}_i(2)\right) \quad \text{and} \quad W_m = W_0^{1,p}(\Omega) \cap V_m^\perp.$$

Then we have

$$W_0^{1,p}(\Omega) = V_m \bigoplus W_m \quad \text{and} \quad d_m = \dim V_m < \infty.$$

In what follows let

$$\partial B_\rho^m = \left\{u \in V_m : \|u\|_{W_0^{1,p}(\Omega)} = \rho\right\}, \rho > 0.$$

Proposition 4.2. *If hypotheses H_2 hold, then we can find $\lambda_0^* \in (0,\lambda^*]$, where $\lambda^* > 0$ is as in Proposition 3.4, such that for all $\lambda \in (0,\lambda_0^*)$ there exists a number $\rho = \rho(\lambda) > 0$ for which*

$$\sup\left[\varphi_\lambda(u) : u \in \partial B_\rho^m\right] < 0.$$

Proof. Let $\eta(x) = f'_s(x,0)$. Given $\varepsilon > 0$, by virtue of hypotheses H_2(i),(iii), there exists a number $c_7 = c_7(\varepsilon) > 0$ such that

$$F(x,s) \geq \frac{1}{2}\left(\eta(x) - \varepsilon\right)s^2 - c_7|s|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{4.1}$$

Taking into account (4.1) we obtain for $u \in V_m$

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(x,u)dx \\
&\le \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{q}\|u\|_q^q - \frac{1}{2}\int_\Omega \eta(x)u^2dx \\
&\quad + \frac{\varepsilon}{2}\|u\|_2^2 + c_7\|u\|_p^p \\
&= \frac{1}{2}\left[\|\nabla u\|_2^2 - \int_\Omega \eta(x)u^2dx + \varepsilon\|u\|_2^2\right] \\
&\quad + \left[\frac{1}{p}\|\nabla u\|_p^p + \frac{\lambda}{q}\|u\|_q^q + c_7\|u\|_p^p\right].
\end{aligned} \tag{4.2}$$

Because of $u \in V_m$ and due to hypothesis H_2(iii) along with Lemma 2.6(b) we verify that

$$\|\nabla u\|_2^2 - \int_\Omega \eta(x)u^2dx \le -\hat{\xi}_1\|u\|^2_{H_0^1(\Omega)}.$$

Since V_m is finite dimensional it is clear that all norms of V_m are equivalent. Therefore, from (4.2) and for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}
\varphi_\lambda(u) &\le -c_8\|u\|^2_{W_0^{1,p}(\Omega)} + c_9\left(\lambda\|u\|^q_{W_0^{1,p}(\Omega)} + \|u\|^p_{W_0^{1,p}(\Omega)}\right) \\
&= \left[-c_8 + c_9\left(\lambda\|u\|^{q-2}_{W_0^{1,p}(\Omega)} + \|u\|^{p-2}_{W_0^{1,p}(\Omega)}\right)\right]\|u\|^2_{W_0^{1,p}(\Omega)}
\end{aligned}$$

for some $c_8 = c_8(\varepsilon), c_9 > 0$.

We consider the function $\hat{\beta}_\lambda(t) = \lambda t^{q-2} + t^{p-2}$ and recall that $q < 2 < p$. As in the proof of Proposition 3.4 we can find $\hat{\lambda}^* > 0$ such that for all $\lambda \in \left(0, \hat{\lambda}^*\right)$ there exists $\rho = \rho(\lambda) > 0$ for which

$$\varphi_\lambda(u) < 0 \quad \text{for all } u \in \partial B_\rho^m.$$

Taking $\lambda_0^* = \min\left\{\hat{\lambda}^*, \lambda^*\right\}$ proves the assertion of the proposition. □

We have another useful result.

Proposition 4.3. *Let hypotheses H_2 be satisfied and let $\lambda > 0$. Then there holds* $\varphi_\lambda\big|_{W_m} \ge 0$.

Proof. Taking into accout (2.2), (2.4) as well as hypothesis H_2(iv) we have for $u \in W_m$

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(x,u)dx \\
&\ge \frac{1}{p}\left[\|\nabla u\|_p^p - \hat{\lambda}_1(p)\|u\|_p^p\right] + \frac{1}{2}\left[\|\nabla u\|_2^2 - \hat{\lambda}_{m+1}(2)\|u\|_2^2\right] \\
&\ge 0.
\end{aligned}$$

□

Now we are ready to prove the complete multiplicity theorem concerning problem $(P)_\lambda$ for all $\lambda > 0$ sufficiently small.

Theorem 4.4. *If hypotheses H_2 hold, then there exists $\lambda_0^* > 0$ such that for all $\lambda \in (0, \lambda_0^*]$ problem $(P)_\lambda$ admits at least five distinct nontrivial solutions*

$$u_0, \hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}, \quad v_0, \hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}, \quad \text{and} \quad y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$$

such that

$$u_0(x), \hat{u}(x) > 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad v_0(x), \hat{v}(x) < 0 \quad \text{for all } x \in \Omega.$$

Proof. As it is always the case in multiplicity theorems, we assume that the energy functional φ_λ has a finite critical set or otherwise we already have a fifth solution and so we are done (recall that the critical points of the energy functional are solutions of our problem). From Proposition 3.6 we know that we can find $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem $(P)_\lambda$ has at least four nontrivial constant sign solutions

$$u_0, \hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\} \quad \text{and} \quad v_0, \hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}$$

such that

$$u_0(x), \hat{u}(x) > 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad v_0(x), \hat{v}(x) < 0 \quad \text{for all } x \in \Omega.$$

From Proposition 3.6 we know that u_0 and v_0 are local minimizers of φ_λ. Hence, due to (2.5),

$$C_k\left(\varphi_\lambda, u_0\right) = C_k\left(\varphi_\lambda, v_0\right) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.3}$$

Furthermore, the proof of Proposition 3.6 shows that $\hat{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$ and $\hat{v} \in -\left(C_0^1(\overline{\Omega})_+\right) \setminus \{0\}$ are critical points of φ_λ^+ and φ_λ^-, respectively, of mountain-pass type such that

$$0 < m_\rho^+ \leq \varphi_\lambda\left(\hat{u}\right) \quad \text{and} \quad 0 < m_\rho^- \leq \varphi_\lambda\left(\hat{v}\right). \tag{4.4}$$

Since φ_λ is coercive (see Proposition 3.2), it is bounded from below. This fact along with Propositions 4.2, 4.3 imply the existence of $\lambda_0^* \in (0, \lambda^*]$ such that for all $\lambda \in (0, \lambda_0^*)$ φ_λ fulfills the assumptions of Theorem 3.1 in Perera [22]. Hence, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\varphi_\lambda}, \quad \varphi_\lambda\left(y_0\right) < 0 = \varphi_\lambda(0), \quad \text{and} \quad C_{d_m-1}\left(\varphi_\lambda, y_0\right) \neq 0. \tag{4.5}$$

From (4.5) it follows that y_0 is a nontrivial solution of $(P)_\lambda$ for all $\lambda \in (0, \lambda_0^*)$. Since $m \geq 3$ we note that $d_m - 1 \geq 2$. Therefore, from (4.3) and (4.5) we conclude that $y_0 \notin \{u_0, v_0\}$ and from (4.4) and (4.5) it follows that $y_0 \notin \{\hat{u}, \hat{v}\}$. Finally, as before, the nonlinear regularity theory implies $y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$. This finishes the proof. □

Remark 4.5. *In contrast to the problems where the concavity enters in the nonlinearity with a positive sign (see Gasiński-Papageorgiou [12] and Hu-Papageorgiou [14]), here we are unable to show that the fifth solution y_0 is nodal. It is an interesting open problem whether y_0 has changing sign. Finally, we mention that we could have used the differential operator $-\Delta_p u - \mu\Delta u$ with $\mu > 0$ without any problem. For simplicity in the presentation we have assumed that $\mu = 1$.*

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(N. S. Papageorgiou) National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
E-mail address: npapg@math.ntua.gr

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
E-mail address: winkert@math.tu-berlin.de