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Entire Extremal Solutions for Elliptic Inclusions of Clarke's Gradient Type

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Abstract. We consider multivalued quasilinear elliptic problems of hemivariational type in all of $\mathbb{R}^N$ given by

$$-\Delta_p u + \partial j(\cdot, u) \ni 0 \quad \text{in } \mathcal{D}',$$

and show the existence of entire extremal solutions by applying the method of sub- and supersolutions without imposing any condition at infinity. Due to the unboundedness of the domain, standard variational methods cannot be applied. The novelty of our approach is on the one hand to obtain entire solutions and on the other hand that Clarke's generalized gradient need only satisfies a natural growth condition. In the last section conditions are provided that ensure the existence of nontrivial positive solutions.
Keywords. Elliptic inclusions, Clarke's generalized gradient, p-Laplacian, Sub-supersolution, Unbounded domains, Fréchet spaces, Locally convex spaces

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1. Introduction

This paper deals with quasilinear elliptic differential inclusions of Clarke's gradient type defined in all of $\mathbb{R}^N$ in the form

$$-\Delta_p u + \partial j(\cdot, u) \ni 0 \quad \text{in } \mathcal{D}', \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u), 1 < p < \infty$, is the negative p-Laplacian and the function $j : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is assumed to be measurable in $x \in \mathbb{R}^N$ for all $s \in \mathbb{R}$, and locally Lipschitz continuous in $s \in \mathbb{R}$ for almost all (a.a.) $x \in \mathbb{R}^N$. The multivalued function $s \mapsto \partial j(x, s)$ stands for Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j(x, s)$ and is given by

$$\partial j(x, s) = \{\xi \in \mathbb{R} : j^o(x, s; r) \geq \xi r, \forall r \in \mathbb{R}\},$$

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for a.a. $x \in \mathbb{R}^N$, where $j^{\mathrm{o}}(x,s;r)$ is the generalized directional derivative of j at s in the direction r defined by

$$j^{\mathrm{o}}(x,s;r) = \limsup_{y \to s, t \downarrow 0} \frac{j(x,y+tr) - j(x,y)}{t},$$

(see [10, Chapter 2]). We denote by $\mathcal{D} = C_0^\infty(\mathbb{R}^N)$ the space of all infinitely differentiable functions with compact support in $\mathbb{R}^N$ and by $\mathcal{D}'$ its dual space.

This type of hemivariational inequalities has been studied by various authors on bounded domains. For Dirichlet boundary conditions under local growth conditions, we refer, e.g., to [8] and for hemivariational inequalities with measure data on the right-hand side see [6]. Single valued problems in the form (1.1) for Neumann boundary conditions of Clarke's gradient type are considered in [5]. In [4] the author discussed our problem (1.1) with a multivalued term in form of a state-dependent subdifferential in all of $\mathbb{R}^N$ which turns out to be a special case of problem (1.1). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider problem (1.1) under Dirichlet boundary values. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. If j is a primitive of f, meaning that

$$j(x,s) := \int_0^s f(x,t)dt,$$

then $s \mapsto j(x,s)$ is continuously differentiable and hence, $\partial j(x,s) = \left\{\frac{\partial j(x,s)}{\partial s}\right\} = \big\{f(x,s)\big\}$. Thus, problem (1.1) is simplified to the elliptic boundary value problem

$$u \in W_0^{1,p}(\Omega) : \quad -\Delta_p u + f(\cdot,u) = 0 \quad \text{in } (W_0^{1,p}(\Omega))',$$

for which the method of sub- and supersolutions is well known (see [7, Chapter 3]). Comparison principles for general elliptic operators A, in particular for the negative p-Laplacian $-\Delta_p$, and Clarke's gradient $s \mapsto \partial j(x,s)$ satisfying a one-sided growth condition in the form

$$\xi_1 \le \xi_2 + c_1(s_2 - s_1)^{p-1}, \tag{1.2}$$

for all $\xi_i \in \partial j(x,s_i), i = 1,2$, for a.a. $x \in \Omega$, and for all s_1, s_2 with $s_1 < s_2$ can also be found in [7, Chapter 4]. Recently, a new comparison result for inclusions of the form (1.1) for bounded domains without the condition (1.2) has been obtained in [9].

The main goal of this paper is to prove the existence of entire extremal solutions for the inclusion (1.1) within a sector of an ordered pair of sub- and supersolutions $\underline{u}, \overline{u}$ without assuming any conditions as in (1.2).

The paper is organized as follows. In Section 2 we formulate our notations and hypotheses and in Section 3 we prove our main result about the existence of extremal solutions. In the end of Section 3 we consider the relation to the problem in [4] and finally, we give an example of the construction of sub- and supersolutions in Section 4.

2. Notations and Hypotheses

Let $\mathcal{W} = W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ be the local Sobolev space of all functions $u : \mathbb{R}^N \to \mathbb{R}$, which belong to the Sobolev space $W^{1,p}(\Omega)$ for every compact domain $\Omega \subset \mathbb{R}^N$. The topology of the locally convex space $\mathcal{W}$ is described by the family of seminorms $\{h_k : k = 1, 2, \ldots\}$ given by $h_k(u) = \|u\|_{W^{1,p}(B_k)}$, where $B_k \subset \mathbb{R}^N$ is the ball of radius k. A sequence $(u_n) \subset \mathcal{W}$ converges to u if and only if

$$h_k(u_n - u) \to 0, \quad \text{as } n \to \infty, \text{ for all } k = 1, 2, \ldots.$$

Since the space $\mathcal{W}$ has a countable fundamental system of seminorms, there exists a metric d on $\mathcal{W}$ for which $(\mathcal{W}, d)$ is a complete metric vector space. Such spaces are called Frechét spaces (see [12, Theorem 25.1, Corollary 25.2]). For fixed k we denote $\mathcal{W}_k = W^{1,p}(B_k)$ and by $i_k : \mathcal{W} \to \mathcal{W}_k$ the mapping defined by $\mathcal{W} \ni u \mapsto u|_{B_k} \in \mathcal{W}_k$, where $u|_{B_k}$ denotes the restriction of u to B_k. Analogously, we define the local Lebesgue space $\mathcal{L}^q := L^q_{\text{loc}}(\mathbb{R}^N)$, where q satisfies the equation $\frac{1}{p} + \frac{1}{q} = 1$. Note that $\mathcal{L}^q$ is equipped with the natural partial ordering $\leq$ defined by $u \leq v$ iff $v - u \in \mathcal{L}^q_+ := L^p_{\text{loc},+}(\mathbb{R}^N)$ which stands for the set of all nonnegative functions of $\mathcal{L}^q$.

Definition 2.1. A function $u \in \mathcal{W}$ is said to be a solution of (1.1), if there exists a function $\gamma \in \mathcal{L}^q$ such that

(i) $\gamma(x) \in \partial j(x, u(x))$, for a.a. $x \in \mathbb{R}^N$,

(ii) $\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} \gamma \varphi dx = 0$, for all $\varphi \in \mathcal{D}$.

Definition 2.2. A function $\underline{u} \in \mathcal{W}$ is said to be a subsolution of (1.1), if there exists a function $\underline{\gamma} \in \mathcal{L}^q$ such that

(i) $\underline{\gamma}(x) \in \partial j(x, \underline{u}(x))$, for a.a. $x \in \mathbb{R}^N$,

(ii) $\int_{\mathbb{R}^N} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi dx + \int_{\mathbb{R}^N} \underline{\gamma} \varphi dx \leq 0$, for all $\varphi \in \mathcal{D}_+$.

Definition 2.3. A function $\overline{u} \in \mathcal{W}$ is said to be a supersolution of (1.1), if there exists a function $\overline{\gamma} \in \mathcal{L}^q$ such that

(i) $\overline{\gamma}(x) \in \partial j(x, \overline{u}(x))$, for a.a. $x \in \mathbb{R}^N$,

(ii) $\int_{\mathbb{R}^N} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi dx + \int_{\mathbb{R}^N} \overline{\gamma} \varphi dx \geq 0$, for all $\varphi \in \mathcal{D}_+$.

Here, $\mathcal{D}_+ := \{\varphi \in \mathcal{D} : \varphi \geq 0\}$ stands for all nonnegative functions of $\mathcal{D}$. In order to formulate our main results we suppose the following hypotheses for the function j and its Clarke's gradient $\partial j(x, \cdot)$ in problem (1.1).

(j1) The mapping $x \mapsto j(x, s)$ is measurable for all $s \in \mathbb{R}$ and $s \mapsto j(x, s)$ is locally Lipschitz continuous for a.a. $x \in \mathbb{R}^N$.

(j2) There is a constant $c > 0$ such that

$$\xi \in \partial j(x, s) : |\xi| \leq c(1 + |s|^{p-1}),$$

for a.a. $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$.

3. Main results

Now, we want to show that the assumptions (j1) and (j2) are sufficient to ensure the existence of entire extremal solutions of (1.1) within the interval $[\underline{u}, \overline{u}]$. Our main result in this paper is given in the following theorem.

Theorem 3.1. *Let conditions* (j1)–(j2) *be satisfied and let* $\underline{u}, \overline{u}$ *be a pair of sub- and supersolutions of problem* (1.1) *satisfying* $\underline{u} \leq \overline{u}$. *Then there exist extremal solutions of* (1.1) *belonging to the interval* $[\underline{u}, \overline{u}]$.

Proof. First we select a sequence of open balls $(B_k) \subset \mathbb{R}^N, k = 1, 2, \dots,$ whose union is equal to $\mathbb{R}^N$, that is, $\bigcup_{k=1}^{\infty} B_k = \mathbb{R}^N$. We construct a sequence $(U_k, \Gamma_k) \subset \mathcal{W} \times \mathcal{L}^q$ as follows: By means of the given supersolution according to Definition 2.3, one defines

$$\begin{aligned} U_0 &:= \overline{u}, \quad U_k(x) = \begin{cases} u_k(x) & \text{for } x \in B_k \\ \overline{u}(x) & \text{for } x \in \mathbb{R}^N \setminus B_k \end{cases} \\ \Gamma_0 &:= \overline{\gamma}, \quad \Gamma_k(x) = \begin{cases} \gamma_k(x) & \text{for } x \in B_k \\ \overline{\gamma}(x) & \text{for } x \in \mathbb{R}^N \setminus B_k, \end{cases} \end{aligned} \tag{3.1}$$

where the pair $(u_k, \gamma_k) \in W^{1,p}(B_k) \times L^q(B_k)$ denotes the greatest solution of the differential inclusion

$$\begin{aligned} -\Delta_p u_k + \partial j(\cdot, u_k) &\ni 0, \quad \text{in } B_k \\ u_k &= \overline{u}, \quad \text{on } \partial B_k \end{aligned} \tag{P_k}$$

in the order interval $[\underline{u}|_{B_k}, \overline{u}|_{B_k}]$. We recall that a pair $(u_k, \gamma_k) \in W^{1,p}(B_k) \times L^q(B_k)$ is a solution of (P_k) if the following holds:

(1) $u_k = \overline{u}$ on ∂B_k,
(2) $\gamma_k(x) \in \partial j(x, u_k(x))$, for a.a. $x \in B_k$,
(3) $\int_{B_k} |\nabla u_k|^{p-2} \nabla u_k \nabla \varphi dx + \int_{B_k} \gamma_k \varphi dx = 0$, for all $\varphi \in C_0^{\infty}(B_k)$.

Obviously, the functions $\underline{u}|_{B_k}, \overline{u}|_{B_k}$ create an ordered pair of sub- and supersolutions to the auxiliary problem (P_k) and the existence of a greatest solution $u_k \in [\underline{u}|_{B_k}, \overline{u}|_{B_k}]$ of (P_k) follows directly from [9, Theorem 4.1, Corollary 4.1]. The extensions (U_k, V_k) of (u_k, v_k) are well-defined and belong to $\mathcal{W} \times \mathcal{L}^q$.

By the construction of U_k one sees immediately that $U_1 \leq U_0$ is true. The function $u_2 \in W^{1,p}(B_2)$ is the greatest solution of (P_2) in the interval $[\underline{u}|_{B_2}, \overline{u}|_{B_2}]$. Furthermore, $u_2|_{B_1}$ is a subsolution of (P_1) in B_1, and $\overline{u}|_{B_1}$ is a supersolution of (P_1) in B_1 satisfying $u_2|_{B_1} \leq \overline{u}|_{B_1}$. Since $u_1 \in W^{1,p}(B_1)$ is the greatest solution of (P_1) in $[\underline{u}|_{B_1}, \overline{u}|_{B_1}] \supset [u_2|_{B_1}, \overline{u}|_{B_1}]$, we obtain $u_2|_{B_1} \leq u_1$ and therefore $U_2 \leq U_1$. In order to generalize this result, we argue per induction and have by definition of U_k that $u_{k+1}|_{B_k}$ is a subsolution of (P_k) and u_k is the greatest solution in $[\underline{u}|_{B_k}, \overline{u}|_{B_k}] \supset [u_{k+1}|_{B_k}, \overline{u}|_{B_k}]$. This yields

$$\underline{u} \leq \cdots \leq U_{k+1} \leq U_k \leq \cdots \leq U_1 \leq U_0 = \overline{u},$$

and consequently,

$$\lim_{k\to\infty} U_k(x) = U^*(x), \quad \text{for almost all } x \in \mathbb{R}^N.$$

To show that U^* belongs to $\mathcal{W}$, let $\Omega \subset \mathbb{R}^N$ be any compact set, which implies the existence of an open ball B_k satisfying $\Omega \subset B_k$. Due to the fact that $\underline{u}, \overline{u}$ generate lower and upper bounds for U_l, we obtain the boundedness of U_l with respect to the norm in $L^p(B_k)$, that is,

$$\|U_l\|_{L^p(B_k)} \leq c_k, \quad \text{for all } l = 1, 2, \ldots, \tag{3.2}$$

where c_k are some positive constants depending only on k. Now we are going to prove the boundedness of ∇U_l in $L^p(B_k)$. One observes that each U_l with $l \geq k+1$ fulfills in B_{k+1}

$$-\Delta_p U_l + \partial j(\cdot, U_l) \ni 0,$$

which by Definition 2.1 means

$$\int_{B_{k+1}} |\nabla U_l|^{p-2} \nabla U_l \nabla \varphi dx + \int_{B_{k+1}} \Gamma_l \varphi dx = 0, \quad \text{for all } \varphi \in C_0^\infty(B_{k+1}), \tag{3.3}$$

where we have

$$\Gamma_l(x) \in \partial(x, U_l(x)), \quad \text{for almost all } x \in \mathbb{R}^N.$$

Since $W_0^{1,p}(B_{k+1})$ is the closure of $C_0^\infty(B_{k+1})$ in $W^{1,p}(B_{k+1})$ (see [1]), the validity of (3.3) for all $\varphi \in W_0^{1,p}(B_{k+1})$ can be proven easily by using completion techniques. With the aid of [11, Theorem 1.2.2] we introduce a function $\vartheta \in \mathcal{D}$ given by the following properties:

1. $0 \leq \vartheta(x) \leq 1$ for all $x \in \mathbb{R}^N$,
2. $\vartheta(x) = 0$ for all $x \in \mathbb{R}^N \setminus B_{k+1}$,
3. $\vartheta(x) = 1$ for all $x \in \overline{B_k}$.

Additionally, it holds

$$\max\left(\sup_{B_{k+1}} \vartheta, \sup_{B_{k+1}} |\nabla \vartheta|^p\right) \leq c, \tag{3.4}$$

where c is a positive constant. By using the special test function $\varphi = U_l \cdot \vartheta^p \in W_0^{1,p}(B_{k+1})$ in the left term of (3.3), one gets along with Young's inequality

$$\begin{aligned}
&\int_{B_{k+1}} |\nabla U_l|^{p-2}\nabla U_l \nabla(U_l\vartheta^p)dx \\
&= \int_{B_{k+1}} \vartheta^p|\nabla U_l|^p dx + p\int_{B_{k+1}} |\nabla U_l|^{p-2}\nabla U_l U_l \vartheta^{p-1}\nabla\vartheta dx \\
&\geq \int_{B_{k+1}} \vartheta^p|\nabla U_l|^p dx - p\int_{B_{k+1}} \varepsilon|\nabla U_l|^p|\vartheta|^p dx - p\int_{B_{k+1}} C(\varepsilon)|U_l|^p|\nabla\vartheta|^p dx \\
&\geq \int_{B_{k+1}} (1-p\varepsilon)\vartheta^p|\nabla U_l|^p dx - p\int_{B_{k+1}} C(\varepsilon)|U_l|^p|\nabla\vartheta|^p dx,
\end{aligned}$$

where ε is selected such that $\varepsilon < \frac{1}{p}$. Applying (j2) along with (3.4) and (3.2) yields

$$\begin{aligned}
\int_{B_{k+1}} (1-p\varepsilon)\vartheta^p|\nabla U_l|^p dx &\leq p\int_{B_{k+1}} C(\varepsilon)|U_l|^p|\nabla\vartheta|^p dx + \int_{B_{k+1}} \Gamma_l U_l \vartheta^p dx \\
&\leq p\int_{B_{k+1}} C(\varepsilon)|U_l|^p|\nabla\vartheta|^p dx + \int_{B_{k+1}} (c + c|U_l|^{p-1})|U_l|\vartheta^p dx \\
&\leq \widetilde{c},
\end{aligned}$$

where $\widetilde{c}$ is a positive constant which depends only on k. The boundedness of the gradient ∇U_l in $L^p(B_k)$ follows directly by the estimate

$$\int_{B_k} |\nabla U_l|^p dx \leq \int_{B_{k+1}} \vartheta^p|\nabla U_l|^p dx \quad \text{for any } l \geq k+1,$$

which implies along with (3.2) that $\|U_l\|_{W^{1,p}(B_k)} \leq \widehat{c}_k$ for all $l = 1, 2, \ldots$. The reflexivity of $W^{1,p}(B_k), 1 < p < \infty$, ensures the existence of a weakly convergent subsequence of U_l. Due to the compact imbedding $W^{1,p}(B_k) \hookrightarrow L^p(B_k)$ and the monotony of U_l we get, for the entire sequence U_l,

$$U_l\,|_{B_k} \rightharpoonup U^*\,|_{B_k} \;\text{ in } W^{1,p}(B_k) \quad \text{and} \quad U_l\,|_{B_k} \to U^*\,|_{B_k} \;\text{ in } L^p(B_k).$$

We have $U^* \in W^{1,p}(B_k)$ and since $\Omega \subset B_k$ it follows $U^* \in W^{1,p}(\Omega)$. As Ω is a freely selected compact domain in $\mathbb{R}^N$, we obtain $U^* \in \mathcal{W}$. Our aim is to show that U^* is the greatest solution of (1.1) in $[\underline{u}, \overline{u}]$. Due to (3.1) it holds

$$\Gamma_k \in \partial j(x, U_k(x)) \quad \text{a.a. in } \mathbb{R}^N \text{ and for all } k. \tag{3.5}$$

Immediately, the boundedness of Γ_k in $\mathcal{L}^q$ is a consequence of condition (j2) and by using the diagonal process of Cantor one shows the existence of a weakly convergent subsequence of (Γ_k), still denoted by Γ_k. In fact, since $\mathcal{L}^q$ is a reflexive Fréchet space for $1 < q < \infty$ (see [12, Theorem 25.15]), we have

$$\int_{\mathbb{R}^N} \Gamma_k \varphi dx \to \int_{\mathbb{R}^N} \Gamma^* \varphi dx \quad \forall \varphi \in \mathcal{D} \text{ as } k \to \infty. \tag{3.6}$$

Due to (3.5) we get, for any ball B_k,

$$\Gamma_l(x) \in \partial j(x, U_l(x)), \quad \text{a.a. } x \in B_k, l = 1, 2, \ldots,$$

which implies

$$\int_{B_k} \Gamma_l \varphi dx \le \int_{B_k} j^o(x, U_l; \varphi) dx, \quad \text{for all } \varphi \in C_0^\infty(B_k).$$

Using Fatou's Lemma and the upper semicontinuity of j^o yields

$$\limsup_{l\to\infty} \int_{B_k} \Gamma_l \varphi dx \le \int_{B_k} \limsup_{l\to\infty} j^o(x, U_l; \varphi) dx \le \int_{B_k} j^o(x, U^*; \varphi) dx,$$

which shows in view of (3.6)

$$\int_{B_k} \Gamma^* \varphi dx \le \int_{B_k} j^o(x, U^*; \varphi) dx, \quad \forall \varphi \in C_0^\infty(B_k). \tag{3.7}$$

We are going to show that (3.7) implies $\Gamma^*(x) \in \partial j(x, U^*(x))$ for a.a. $x \in B_k$. The mapping $r \mapsto j^o(x, s; r)$ is positively homogeneous and inequality (3.7) holds, in particular, for all $\varphi \in C_0^\infty(B_k)_+$. We obtain

$$\int_{B_k} \Gamma^* \varphi dx \le \int_{B_k} j^o(x, U^*; 1) \varphi dx, \quad \forall \varphi \in C_0^\infty(B_k)_+.$$

By [10, Proposition 2.1.2] Clarke's generalized directional derivative j^o fulfills

$$j^o(x, s; r) = \max\{\xi r : \xi \in \partial j(x, s)\},$$

and since $\partial j(x, s)$ is a nonempty, convex, and compact subset of $\mathbb{R}$, there exists a function $\Gamma_1^* : B_k \to \mathbb{R}$ such that

$$j^o(x, U^*(x); 1) = \Gamma_1^*(x), \quad \text{for a.a. } x \in B_k, \tag{3.8}$$

where

$$\Gamma_1^*(x) = \max\{\xi : \xi \in \partial j(x, U^*(x))\}. \tag{3.9}$$

Applying the general approximation results in [3] for lower (respectively, upper) semicontinuous functions in Hilbert spaces yields a sequence of locally Lipschitz functions converging pointwise to j^o. This implies that $s \mapsto j^o(x, s; 1)$ is superpositionally measurable, meaning that the mapping $x \mapsto j^o(x, u(x); 1)$ is measurable for all measurable functions $u : B_k \to \mathbb{R}$. Due to (3.8) and (j2) we infer $\Gamma_1^* \in L^q(B_k)$. Using (3.7) proves

$$\int_{B_k} \Gamma^* \varphi dx \le \int_{B_k} \Gamma_1^* \varphi dx, \ \forall \varphi \in C_0^\infty(B_k)_+,$$

which implies

$$\Gamma^*(x) \le \Gamma_1^*, \quad \text{for a.a. } x \in B_k. \tag{3.10}$$

Testing (3.7) with nonpositve functions $\varphi = -\psi$, where $\psi \in C_0^\infty(B_k)_+$, we have

$$-\int_{B_k} \Gamma^* \psi dx \le \int_{B_k} j^o(x, U^*; -1)\psi dx, \quad \forall \psi \in C_0^\infty(B_k)_+. \tag{3.11}$$

The same arguments as above yield the existence of a function $\tau \in L^q(B_k)$ such that

$$\tau(x) = \max\{-\xi : \xi \in \partial j(x, U^*(x))\} = -\min\{\xi : \xi \in \partial j(x, U^*(x))\}, \tag{3.12}$$

which implies by setting $\Gamma_2^* = -\tau$ in (3.11) that

$$-\int_{B_k} \Gamma^* \psi dx \le -\int_{B_k} \Gamma_2^* \psi dx, \quad \forall \psi \in C_0^\infty(B_k)_+,$$

and therefore one gets

$$\int_{B_k} \Gamma^* \psi dx \ge \int_{B_k} \Gamma_2^* \psi dx, \quad \forall \psi \in C_0^\infty(B_k)_+.$$

From the last inequality we infer

$$\Gamma^*(x) \ge \Gamma_2^*, \quad \text{for a.a. } x \in B_k. \tag{3.13}$$

In view of (3.9), (3.10), (3.12), (3.13) and $\Gamma_2^* = -\tau$ we see at once that

$$\Gamma^*(x) \in \partial j(x, U^*(x)) \quad \text{for a.a. } x \in B_k. \tag{3.14}$$

Let $\varphi \in \mathcal{D}$ be arbitrary fixed. Then there exists an index k such that the support of φ fulfills $\operatorname{supp} \varphi \subset B_k$. The approximations above yield, for any $l \ge k$

$$\int_{\mathbb{R}^N} |\nabla U_l|^{p-2} \nabla U_l \nabla \varphi dx + \int_{\mathbb{R}^N} \Gamma_l \varphi dx = 0,$$

or equivalently

$$\int_{B_k} |\nabla U_l|^{p-2} \nabla U_l \nabla \varphi dx + \int_{B_k} \Gamma_l \varphi dx = 0. \tag{3.15}$$

It is well known that $-\Delta_p : W^{1,p}(B_k) \to (W^{1,p}(B_k)^*)$ is continuous, bounded, and pseudomonotone for $1 < p < \infty$. We have $U_l \rightharpoonup U^*$ in $W^{1,p}(B_k)$ and due to the pseudomonotonicity it holds $-\Delta_p U_l \rightharpoonup -\Delta_p U^*$ in $(W^{1,p}(B_k))^*$. Along with the weak convergence of Γ_l in $L^q(B_k)$ we can pass to the limit in (3.15) and obtain

$$\int_{\mathbb{R}^N} |\nabla U^*|^{p-2} \nabla U^* \nabla \varphi dx + \int_{\mathbb{R}^N} \Gamma^* \varphi dx = 0. \tag{3.16}$$

The statements in (3.14) and (3.16) show that the pair (U^*, Γ^*) is a solution of the problem (1.1) in $[\underline{u}, \overline{u}]$. In order to complete the proof we have to prove that U^* is the greatest solution of (1.1) in $[\underline{u}, \overline{u}]$. Let $\widetilde{u}$ be any solution of (1.1) in the order interval $[\underline{u}, \overline{u}]$. Obviously, the solution $\widetilde{u}$ is also a subsolution of (1.1), which implies by the construction in (3.1) that the inequality $\widetilde{u} \le U_l \le \overline{u}$ is valid for all $l = 1, 2, \ldots$. This yields $\widetilde{u} \le U_l$, which shows that U^* must be the greatest solution of (1.1) in $[\underline{u}, \overline{u}]$. In the same way one can show the existence of a smallest solution. □

Remark 3.2. Notice that Theorem 3.1 can be extended for problems of the form

$$Au + \partial j(\cdot, u) \ni 0 \ \text{ in } \mathcal{D}',$$

where

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)), \quad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \right) \tag{3.17}$$

is a general operator of the Leray–Lions type. The proof in this case can be shown by using similar arguments.

Remark 3.3. The elliptic inclusion problem with state-dependent subdifferentials investigated by Carl in [4] has the form

$$Au + \beta(\cdot, u, u) \ni 0 \ \text{ in } \mathcal{D}',$$

where A is a general operator of the Leray–Lions type like in (3.17) and $\beta(x, u, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ is a maximal monotone graph in $\mathbb{R}^2$ depending continuously on the unknown u. The multifunction β is generated by $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies the following conditions:

(f1) $(x, r) \to f(x, r, s)$ is a Carathéodory function uniformly with respect to s, which means that f is measurable in x for all $(r, s) \in \mathbb{R} \times \mathbb{R}$ and continuous in r for a.a. $x \in \mathbb{R}^N$ uniformly with respect to s.

(f2) $s \to f(x, r, s)$ is nondecreasing (possibly discontinuous) for a.a. $x \in \mathbb{R}^N$ and for each $r \in \mathbb{R}$, and it is related to the maximal monotone graph β by

$$\beta(x, r, s) = [f(x, r, s - 0), f(x, r, s + 0)],$$

where $f(x, r, s \pm 0) = \lim_{\varepsilon \downarrow 0} f(x, r, s \pm \varepsilon)$.

(f3) $(x, s) \to f(x, r, s)$ is measurable in $\mathbb{R}^N \times \mathbb{R}$ for each $r \in \mathbb{R}$.

(f4) For a given pair of sub- and supersolutions $\underline{u}, \overline{u}$ satisfying $\underline{u} \le \overline{u}$, there exists a function $k \in \mathcal{L}^q_+$ and a constant $\alpha > 0$ such that

$$|f(x, r, s)| \le k(x),$$

for a.a. $x \in \mathbb{R}^N$ and for all $r \in [\underline{u}(x), \overline{u}(x)]$ and $s \in [\underline{u}(x) - \alpha, \overline{u}(x) + \alpha]$.

The function f is continuous in the second argument and nondecreasing (possibly discontinuous) in the third argument. Thus, $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R})$ and we can set

$$j(x, s) = \int_0^s f(x, t, t)dt,$$

which yields that the function $s \mapsto j(x, s)$ is locally Lipschitz and Clarke's generalized gradient can be represented by $\partial j(x, s) = \beta(x, s, s)$ (for more details see, e.g., [9]). Hence, our paper extends the results in [4] for more general multifunction in form of Clarke's generalized gradients in all of $\mathbb{R}^N$.

4. Construction of Sub- and Supersolutions

In this section we give some conditions to find a pair of sub- and supersolutions of our problem (1.1). The main idea is to use the eigenvalues and the corresponding eigenfunctions of the p-Laplacian on bounded domains with Dirichlet boundary values. We denote by λ_1 the first eigenvalue of the p-Laplacian on the ball B_r with radius r corresponding to its eigenfunction φ_1. This means, φ_1 satisfies the equation

$$\begin{aligned} -\Delta_p u &= \lambda_1 |u|^{p-2} u && \text{in } B_r, \\ u &= 0 && \text{on } \partial B_r. \end{aligned} \tag{4.1}$$

In view of the results of Anane in [2], it is well known that λ_1 is positive and $\varphi_1 \in \operatorname{int}(C^1_0(\overline{B_r})_+)$, where the interior of the positive cone $C^1_0(\overline{B_r})_+$ is given by

$$\operatorname{int}(C^1_0(\overline{B_r})_+) = \Big\{ u \in C^1_0(\overline{B_r}) : u(x) > 0, \forall x \in B_r, \text{ and } \frac{\partial u}{\partial n}(x) < 0, \forall x \in \partial B_r \Big\}.$$

Now we suppose the hypotheses on Clarke's generalized gradient as follows:

(j3) There exists a Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, which fulfills

$$\xi \le g(x,s), \quad \forall s \in \mathbb{R}, \text{ for a.a. in } \mathbb{R}^N, \text{ and for all } \xi \in \partial j(x,s) \tag{4.2}$$

and has the property

$$\liminf_{s\to+0} \left(-\frac{g(x,s)}{s^{p-1}} \right) > \lambda_1, \tag{4.3}$$

uniformly with respect to a.a. $x \in \mathbb{R}^N$. Furthermore, there exists $\widetilde{s} > 0$ such that

$$\partial j(x,\widetilde{s}) \ge 0, \quad \text{for a.a. } x \in \mathbb{R}^N. \tag{4.4}$$

Proposition 4.1. *Let the conditions* (j1)–(j3) *be satisfied. Then there exists a positive ordered pair of sub- and supersolutions*

$$\underline{u}(x) = \begin{cases} \varepsilon\varphi_1(x) & \text{if } x \in B_r \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_r, \end{cases} \qquad \overline{u}(x) = \widetilde{s}, \quad \text{for a.a. } x \in \mathbb{R}^N \tag{4.5}$$

of problem (1.1) *provided that* $\varepsilon > 0$ *is sufficiently small.*

Proof. The eigenfunction φ_1 of (4.1) belongs to $\mathrm{int}(C_0^1(\overline{B_r})_+)$, that means in particular, the outer normal derivative $\frac{\partial \varphi_1}{\partial n}$ on ∂B_r has a negative sign. By applying the Divergence Theorem we have for $\varphi \in \mathcal{D}_+$

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi dx &= \int_{B_r} |\nabla(\varepsilon\varphi_1)|^{p-2} \nabla(\varepsilon\varphi_1) \nabla\varphi dx \\
&= \int_{\partial B_r} |\nabla(\varepsilon\varphi_1)|^{p-2} (\partial(\varepsilon\varphi_1)/\partial\nu) \varphi dx + \int_{B_r} \lambda_1 (\varepsilon\varphi_1)^{p-1} \varphi dx \\
&\le \int_{B_r} \lambda_1 (\varepsilon\varphi_1)^{p-1} \varphi dx \\
&= \int_{\mathbb{R}^N} \lambda_1 \underline{u}^{p-1} \varphi dx.
\end{aligned}$$

This calculation along with (4.2) and (4.3) yields for $\underline{\gamma} \in \partial j(\cdot, \varepsilon\varphi_1)$

$$-\Delta_p(\varepsilon\varphi_1) + \underline{\gamma} \le \lambda_1 (\varepsilon\varphi_1)^{p-1} + g(\cdot, \varepsilon\varphi_1) \le 0,$$

assumed ε is sufficiently small. Due to (4.4) it follows directly that $\overline{u} = \widetilde{s}$ is a positive constant supersolution of (1.1). Choosing ε small enough such that $\underline{u} \le \overline{u}$ completes the proof. □

Proposition 4.1 ensures under the additionally hypothesis (j3) the existence of a positive nontrivial solution u of (1.1) belonging to the order interval of sub- and supersolutions given in (4.5).

Example 4.2. Let $\lambda > \lambda_1$ be fixed and let $j(x,\cdot) : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function satisfying (j1) given by

$$j(x,s) = \begin{cases} -\lambda e^{s-2} - \lambda s - \operatorname{sgn}(s)\dfrac{|x|+2}{p(|x|+1)}|s|^p, & \text{if } s \le 2 \\ -\dfrac{1}{2}\lambda s^2 + 4\lambda s - 9\lambda - \dfrac{|x|+2}{p(|x|+1)}s^p, & \text{if } 2 \le s \le 3 \\ -\lambda e^{-s+3} + \lambda s - \dfrac{7}{2}\lambda - \dfrac{|x|+2}{p(|x|+1)}s^p, & \text{if } s \ge 3. \end{cases}$$

Its generalized Clarke's gradient has the form

$$\partial j(x,s) = \begin{cases} -\lambda e^{s-2} - \lambda - \dfrac{|x|+2}{|x|+1}|s|^{p-1}, & \text{if } s < 2 \\ \left[-2\left(\lambda + \dfrac{|x|+2}{|x|+1}2^{p-2}\right), 2\left(\lambda - \dfrac{|x|+2}{|x|+1}2^{p-2}\right)\right], & \text{if } s = 2 \\ -\lambda s + 4\lambda - \dfrac{|x|+2}{|x|+1}s^{p-1}, & \text{if } 2 < s < 3 \\ \left[\lambda - \dfrac{|x|+2}{|x|+1}3^{p-1}, 2\lambda - \dfrac{|x|+2}{|x|+1}3^{p-1}\right], & \text{if } s = 3 \\ \lambda e^{-s+3} + \lambda - \dfrac{|x|+2}{|x|+1}s^{p-1}, & \text{if } s > 3. \end{cases}$$

One easily verifies that $\partial j(x,\cdot)$ satisfies the condition (j2) and is bounded above by a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ defined as

$$g(x,s) = \begin{cases} |s| - \dfrac{|x|+2}{|x|+1}|s|^{p-1}, & \text{if } s \le 0 \\ -\left(\lambda + \dfrac{|x|+2}{|x|+1}\right)s^{p-1}, & \text{if } 0 \le s \le 1 \\ 3\lambda s - 4\lambda - \dfrac{|x|+2}{|x|+1}s^{p-1}, & \text{if } 1 \le s \le 2 \\ s + 2(\lambda - 1) - \dfrac{|x|+2}{|x|+1}s^{p-1}, & \text{if } s \ge 2. \end{cases}$$

Since g fulfills property (4.3), there exists a positive pair of sub- and supersolutions given by (4.5) and thus, we obtain a nontrivial positive solution $u \in [\underline{u}, \overline{u}]$ of problem (1.1).

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