Nonlinear Differ. Equ. Appl. 17 (2010), 289–302

1021-9722/10/030289-14
published online January 12, 2010
DOI 10.1007/s00030-009-0054-5

Nonlinear Differential Equations and Applications NoDEA

L^∞-Estimates for nonlinear elliptic Neumann boundary value problems

Patrick Winkert

Abstract. In this paper we prove the L^∞-boundedness of solutions of the quasilinear elliptic equation

$$\begin{aligned} Au &= f(x,u,\nabla u) && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x,u) && \text{on } \partial\Omega, \end{aligned}$$

where A is a second order quasilinear differential operator and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ as well as $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying natural growth conditions. Our main result is given in Theorem 4.1 and is based on the Moser iteration technique along with real interpolation theory. Furthermore, the solutions of the elliptic equation above belong to $C^{1,\alpha}(\overline{\Omega})$, if g is Hölder continuous.

Mathematics Subject Classification (2000). 35B45, 35J25, 35J60.

Keywords. A priori estimates, Neumann boundary values, Nonlinear elliptic equations.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the quasilinear elliptic equation

$$\begin{aligned} Au &= f(x,u,\nabla u) && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative of u. Here, A is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)), \tag{1.2}$$

and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ as well as $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions. For $u \in W^{1,p}(\Omega)$ defined on the boundary $\partial\Omega$, we make use of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is well known to be compact. For easy readability we will drop the notation γu and write for short u, respectively, $g(x, u) := g(x, \gamma u)$.

The main goal of this paper is to prove a priori estimates for solutions of the nonlinear elliptic equation in (1.1). For this purpose, we use some important tools like the Moser iteration technique and real interpolation theory. By an a priori estimate, we mean an estimate of the form

$$\|u\|_{L^\infty(\Omega)} \le C,$$

for all possible solutions of (1.1) with some constant C independent of u. Concerning a priori bounds for elliptic equations with zero Neumann conditions we refer to the results in [17] and [19], where they consider problems of the form

$$\begin{aligned} -\Delta u + \lambda u &= f(u),\ u > 0 \quad && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary and $\lambda > 0$. Motreanu et al. have applied the Moser iteration, too, in [11, Proof of Proposition 2.5] to prove L^∞-boundedness for solutions of the Neumann problem

$$\begin{aligned} -\operatorname{div} \vartheta_\varepsilon(z, \nabla v_\varepsilon) &= f_0(z, v_\varepsilon) + \lambda_\varepsilon f_0(z, x_0) - \lambda_\varepsilon |v_\varepsilon - x_0|^{p-2}(v_\varepsilon - x_0) \quad && \text{in } Z, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial Z, \end{aligned}$$

where

$$\vartheta_\varepsilon(z, \xi) = |\xi|^{p-2}\xi + \lambda_\varepsilon |\nabla x_0|^{p-2}\nabla x_0 + \lambda_\varepsilon |\xi - \nabla x_0|^{p-2}(\xi - \nabla x_0),$$

with $Z \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial Z, 0 < \lambda_\varepsilon \le 1, \varepsilon \in (0, 1]$, $x_0 \in L^\infty(\Omega)$ fixed and with a Carathéodory function $f_0 : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfying some growth condition.

The novelty of our paper is the demonstration of a priori estimates for nonlinear elliptic equations with nonlinear nonhomogenous Neumann boundary values of the form (1.1), where the Carathéodory functions f and g depend on $u, \nabla u$ and u, respectively, satisfying a natural growth condition. Additionally, we extend our results and show that every solution of (1.1) belongs to $C^{1,\alpha}(\overline{\Omega})$ in case g satisfies the condition

$$|g(x_1, s_1) - g(x_2, s_2)| \le L \left[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha \right],$$

for all pairs $(x_1, s_1), (x_2, s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where M_0 is a positive constant and $\alpha \in (0, 1]$. The C^1-regularity follows directly from the L^∞-boundedness along with the results of Liebermann in [10].

2. Notations and hypotheses

Let $\frac{1}{p}+\frac{1}{q}=1$. We suppose the following conditions on the operator A and the nonlinearities $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$.

(A1) Each $a_i(x,s,\xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x\in\Omega$ for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and continuous in (s,ξ) for a.e. $x\in\Omega$. Furthermore, a constant $c_0>0$ and a function $k_0\in L^q(\Omega)$ exist so that

$$|a_i(x,s,\xi)|\le k_0(x)+c_0\left(|s|^{p-1}+|\xi|^{p-1}\right), \tag{2.1}$$

for a.e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ.

(A2) The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^N(a_i(x,s,\xi)-a_i(x,s,\xi'))(\xi_i-\xi_i')>0, \tag{2.2}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$, and for all $\xi,\xi'\in\mathbb{R}^N$ with $\xi\neq\xi'$.

(A3) A constant $c_1>0$ and a function $k_1\in L^\infty(\Omega)$ exist such that

$$\sum_{i=1}^N a_i(x,s,\xi)\xi_i\ge c_1|\xi|^p-k_1(x), \tag{2.3}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$, and for all $\xi\in\mathbb{R}^N$.

(F1) $x\mapsto f(x,s,\xi)$ is measurable in Ω for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

(F2) $(s,\xi)\mapsto f(x,s,\xi)$ is continuous in $\mathbb{R}\times\mathbb{R}^N$ for almost all $x\in\Omega$.

(F3) There exists a constant $c_2>0$ such that

$$|f(x,s,\xi)|\le c_2\left(1+|s|^{p-1}+|\xi|^{p-1}\right), \tag{2.4}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^N$.

(G1) $x\mapsto g(x,s)$ is measurable in $\partial\Omega$ for all $s\in\mathbb{R}$.

(G2) $s\mapsto g(x,s)$ is continuous in $\mathbb{R}$ for almost all $x\in\partial\Omega$.

(G3) There exists a constant $c_3>0$ such that

$$|g(x,s)|\le c_3(1+|s|^{p-1}), \tag{2.5}$$

for a.e. $x\in\partial\Omega$ and for all $s\in\mathbb{R}$.

Condition (A1) implies that $A:W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$ is bounded continuous and along with (A2) it holds that A is pseudomonotone. Due to (A1) the operator A generates a mapping from $W^{1,p}(\Omega)$ into its dual space defined by

$$\langle Au,\varphi\rangle=\int_\Omega\sum_{i=1}^N a_i(x,u,\nabla u)\frac{\partial\varphi}{\partial x_i}dx, \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$. Assumption (A3) is a coercivity type condition. The conditions (F3) and (G3) ensure that the corresponding Nemytskij operators $F:W^{1,p}(\Omega)\to L^q(\Omega)$ and $G:L^p(\partial\Omega)\to L^q(\partial\Omega)$ defined by

$$Fu(x)=f(x,u(x),\nabla u(x)),\quad Gu(x)=g(x,u(x)), \tag{2.7}$$

are bounded and continuous (see e.g. [18]). The definition of a solution of problem (1.1) in the weak sense is defined as follows.

Definition 2.1. A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (1.1) if the following holds

$$\int_\Omega \sum_{i=1}^N a_i(x,u,\nabla u)\frac{\partial \varphi}{\partial x_i}dx = \int_\Omega f(x,u,\nabla u)\varphi dx + \int_{\partial\Omega} g(x,u)\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega).$$

Remark 2.2. The growth conditions on the function f and g can be relaxed, replacing $|s|^{p-1}$ by $|s|^q$ for a suitable $q > p-1$. Thanks to the Sobolev embedding and to the trace embedding, the definition of a weak solution to the Neumann problem would also be consistent in this case. For reasons of simplification, we deal with the given growth conditions as in (2.4) and (2.5).

3. The spaces B^s_{pq} and F^s_{pq}

In this section, we give a brief overview about Besov spaces (respectively, Lizorkin–Triebel spaces) which are needed in the proof of our main theorem.

If $\mathcal{A}$ is a Banach space, then

$$l^\sigma_p(\mathcal{A}) = \left\{ \xi : \xi = \{\xi_j\}_{j=0}^\infty ; \xi_j \in \mathcal{A}; \|\xi\|_{l^\sigma_p} = \left(\sum_{j=0}^\infty 2^{j\sigma p}\|\xi_j\|^p_{\mathcal{A}} \right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \leq p < \infty$ and

$$l^\sigma_\infty(\mathcal{A}) = \left\{ \xi : \xi = \{\xi_j\}_{j=0}^\infty ; \xi_j \in \mathcal{A}; \|\xi\|_{l^\sigma_\infty} = \sup_j 2^{j\sigma}\|\xi_j\|_{\mathcal{A}} < \infty \right\}$$

for $p = \infty$ are also Banach spaces (cf. [14, Section 1.18]) where σ is a real number. We recall that $S = S(\mathbb{R}^N)$ is the set of all complex-valued rapidly decreasing infinitely differentiable functions defined on the N-dimensional real Euclidean space $\mathbb{R}^N$. The spaces $S(\mathbb{R}^N)$ and $S'(\mathbb{R}^N)$ (dual space) have their usual topologies, where $S'(\mathbb{R}^N)$ is equipped with the strong topology. We denote by $\mathcal{F}$ the Fourier transform in S and the support of a distribution f is written as $\operatorname{supp} f$. Further, we set

$$M_j = \left\{ \xi : \xi \in \mathbb{R}^N, 2^{j-1} \leq |\xi| \leq 2^{j+1} \right\}, \quad j = 1, 2, \ldots,$$
$$M_0 = \left\{ \xi : \xi \in \mathbb{R}^N, |\xi| \leq 2 \right\}.$$

Then we introduce the spaces $B^s_{pq}(\mathbb{R}^N)$ and $F^s_{pq}(\mathbb{R}^N)$ as follows (see [14, Definition 2.3.1/1]).

Definition 3.1. (a) For $-\infty < s < \infty, 1 < p < \infty$, and $1 \le q < \infty$ one sets

$$B^s_{pq}(\mathbb{R}^N) = \Bigg\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^{\infty} a_j(x); \operatorname{supp} \mathcal{F}a_j \subset M_j;$$
$$\|\{a_j\}\|_{l^s_q(L_p)} = \left[\sum_{j=0}^{\infty} (2^{sj} \|a_j(x)\|_{L_p})^q \right]^{\frac{1}{q}} < \infty \Bigg\}$$

and for $-\infty < s < \infty, 1 < p < \infty$, and $q = \infty$ one sets

$$B^s_{p\infty}(\mathbb{R}^N) = \Bigg\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^{\infty} a_j(x); \operatorname{supp} \mathcal{F}a_j \subset M_j;$$
$$\|\{a_j\}\|_{l^s_\infty(L_p)} = \sup_j 2^{sj} \|a_j(x)\|_{L_p} < \infty \Bigg\}.$$

Further, for $-\infty < s < \infty, 1 < p < \infty$ and $1 \le q \le \infty$ let

$$\|f\|_{B^s_{pq}(\mathbb{R}^N)} = \inf_{f = \sum a_j} \|\{a_j\}\|_{l^s_q(L_p)}.$$

(b) For $-\infty < s < \infty, 1 < p < \infty$, and $1 < q < \infty$ one sets

$$F^s_{pq}(\mathbb{R}^N) = \Bigg\{ f : f \in S'(\mathbb{R}^N); f = \sum_{j=0}^{\infty} a_j(x); \operatorname{supp} \mathcal{F}a_j \subset M_j;$$
$$\|\{a_j\}\|_{L_p(l^s_q)} = \left[\int_\Omega \left(\sum_{j=0}^{\infty} 2^{sjq} |a_j(x)|^q \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} < \infty \Bigg\}.$$

Further, let

$$\|f\|_{F^s_{pq}(\mathbb{R}^N)} = \inf_{f = \sum a_j} \|\{a_j\}\|_{L_p(l^s_q)}.$$

(c) For $-\infty < s < \infty$ and $1 < p < \infty$ one sets

$$H^s_p(\mathbb{R}^N) = F^s_{p2}(\mathbb{R}^N).$$

(d) For $1 < p < \infty$ one sets

$$W^s_p(\mathbb{R}^N) = \begin{cases} H^s_p(\mathbb{R}^N) & \text{if } s = 0, 1, 2, \dots, \\ B^s_{pp}(\mathbb{R}^N) & \text{if } 0 < s \neq \text{ integer}. \end{cases}$$

The proof of the following theorem can be found in [14, Theorem 2.3.2].

Theorem 3.2. 1. *Let $-\infty < s < \infty, 1 < p < \infty$, and $1 \le q \le \infty$. Then $B^s_{pq}(\mathbb{R}^N)$ is a Banach space.*

2. *Let $-\infty < s < \infty, 1 < p < \infty$, and $1 < q < \infty$. Then $F^s_{pq}(\mathbb{R}^N)$ is a Banach space.*

It is clear, that all notations above hold true if we replace $\mathbb{R}^N$ by a bounded domain $\Omega \subset \mathbb{R}^N$. The spaces B^s_{pq} and F^s_{pq} are called Besov and Lizorkin–Triebel spaces, respectively, which are equal in case $p = q$ with $1 < p < \infty$ and $-\infty < s < \infty$. We see that the spaces W^s_p with $s = 1, 2, 3, \dots$ coincide with the well-known Sobolev spaces introduced by S.L.Sobolev and the extension of the definition of the spaces W^s_p to values $0 < s \neq$ integer are the so-called Slobodeckij spaces. Finally, it was shown that H^s_p with $s > 0$ coincide with the well-known Lebesgue (or Liouville, or Bessel-potential) spaces. For more details we refer for example to the books of Triebel in [14–16] or to the monograph of Runst and Sickel in [12].

In our considerations, we need the following continuous embeddings

$$\begin{aligned} &T_1 : B^s_{pp}(\Omega) \to B^{s-\frac{1}{p}}_{pp}(\partial\Omega), \quad \text{with } s > \frac{1}{p},\\ &T_2 : B^{s-\frac{1}{p}}_{pp}(\partial\Omega) = F^{s-\frac{1}{p}}_{pp}(\partial\Omega) \to F^0_{p2}(\partial\Omega) = L^p(\partial\Omega), \quad \text{with } s > \frac{1}{p}, \end{aligned}$$

where Ω is a bounded C^∞-domain (see [12, Page 75 and Page 82], [14, 2.3.1 and 2.3.2] and [15, 3.3.1]). Let $s = m + \iota$ with $m \in \mathbb{N}_0$ and $0 \leq \iota < 1$. Then the embeddings are also valid if $\partial\Omega \in C^{m,1}$ [13]. In [3, Satz 9.40] the proof is given for $p = 2$, however, it can be extended to $p \in (1, \infty)$ by using the Fourier transformation in $L^p(\Omega)$ [4].

We set $s = \frac{1}{p} + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon} > 0$ is arbitrarily fixed such that $s = \frac{1}{p} + \widetilde{\varepsilon} < 1$. Thus, the embeddings are valid for a Lipschitz boundary $\partial\Omega$. This yields

$$T_3 : B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega) \to L^p(\partial\Omega),$$

which means

$$\|v\|_{L^p(\partial\Omega)} \leq c_4 \|v\|_{B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega)}, \quad \forall v \in B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega), \tag{3.1}$$

where c_4 is a positive constant. The real interpolation theory implies

$$\left(F^0_{p2}(\Omega), F^1_{p2}(\Omega)\right)_{\frac{1}{p}+\widetilde{\varepsilon},p} = \left(L^p(\Omega), W^{1,p}(\Omega)\right)_{\frac{1}{p}+\widetilde{\varepsilon},p} = B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega),$$

(see [1,14,15], [16, Section 1.6.2 and 1.6.7]) which ensures the norm estimate

$$\|v\|_{B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega)} \leq c_5 \|v\|^{\frac{1}{p}+\widetilde{\varepsilon}}_{W^{1,p}(\Omega)} \|v\|^{1-\frac{1}{p}-\widetilde{\varepsilon}}_{L^p(\Omega)}, \quad \forall v \in W^{1,p}(\Omega), \tag{3.2}$$

(cf. [14, Theorem 1.3.3 (g)]) with a positive constant c_5 only depending on the boundary $\partial\Omega$.

4. Main results

Theorem 4.1. *Let the conditions (A1)–(A3), (F1)–(F3) and (G1)–(G3) be satisfied. Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1). Then $u \in L^\infty(\Omega)$.*

Proof. To prove the L^∞−regularity of u, we will use the Moser iteration technique (see e.g. [5–9]). It suffices to consider the proof in case $1 \leq p \leq N$, otherwise we would be done. First we are going to show that $u^+ = \max\{u, 0\}$

belongs to $L^\infty(\Omega)$. For $M > 0$ we define $v_M(x) = \min\{u^+(x), M\}$. Letting $K(t) = t$ if $t \le M$ and $K(t) = M$ if $t > M$, it follows by [9, Theorem B.3] that $K \circ u^+ = v_M \in W^{1,p}(\Omega)$ and hence $v_M \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. For real $k \ge 0$ we choose $\varphi = v_M^{kp+1}$, then $\nabla\varphi = (kp+1)v_M^{kp}\nabla v_M$ and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Notice that $u(x) \le 0$ implies directly $v_M(x) = 0$. Testing the weak formulation in Definition 2.1 with $\varphi = v_M^{kp+1}$, one gets

$$\begin{aligned}(kp+1)&\int_\Omega \sum_{i=1}^N a_i(x,u^+,\nabla u^+)v_M^{kp}\frac{\partial v_M}{\partial x_i}dx\\ &= \int_\Omega f(x,u^+,\nabla u^+)v_M^{kp+1}dx + \int_{\partial\Omega} g(x,u^+)v_M^{kp+1}d\sigma. \end{aligned}\tag{4.1}$$

Applying condition (A3) and the Hölder inequality, the left-hand side of (4.1) can be estimated to obtain

$$\begin{aligned}(kp+1)&\int_\Omega \sum_{i=1}^N a_i(x,u^+,\nabla u^+)v_M^{kp}\frac{\partial v_M}{\partial x_i}dx\\
&= (kp+1)\int_\Omega \sum_{i=1}^N a_i(x,v_M,\nabla v_M)\frac{\partial v_M}{\partial x_i}v_M^{kp}dx\\
&\ge (kp+1)\int_\Omega (c_1|\nabla v_M|^p - k_1)v_M^{kp}dx\\
&\ge c_1\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx - e_1(kp+1)\int_\Omega (u^+)^{kp}dx\\
&\ge c_1\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx\\
&\quad - e_1(kp+1)|\Omega|^{\frac{1}{k+1}}\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}},\end{aligned}\tag{4.2}$$

where $e_1 = \|k_1\|_\infty$. The assumption (F3) along with the Hölder inequality and Young's inequality implies

$$\begin{aligned}\int_\Omega & f(x,u^+,\nabla u^+)v_M^{kp+1}dx\\
&\le c_2\int_\Omega (1+|u^+|^{p-1}+|\nabla u^+|^{p-1})v_M^{kp+1}dx\\
&\le c_2|\Omega|^{\frac{p-1}{(k+1)p}}\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + c_2\int_\Omega (u^+)^{(k+1)p}dx\\
&\quad +c_2\int_\Omega \delta|\nabla u^+|^{(p-1)q}v_M^{k(p-1)q}dx + c_2\int_\Omega C(\delta)v_M^{(k+1)p}dx\\
&\le e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx\\
&\quad +\frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx.\end{aligned}\tag{4.3}$$

The same arguments for the boundary integral provide

$$\begin{aligned}\int_{\partial\Omega} g(x,u^+)v_M^{kp+1}d\sigma &\le c_3\int_{\partial\Omega}(1+|u^+|^{p-1})v_M^{kp+1}d\sigma\\ &\le e_3\left(\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\\ &\quad + e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma. \end{aligned} \tag{4.4}$$

Applying the estimates (4.2)–(4.4) to (4.1) one gets

$$\begin{aligned}&\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx\\ &\le e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx\\ &\quad +\frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla(u^+)^{k+1}|^p dx + e_3\left(\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\\ &\quad +e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_5(kp+1)\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}. \end{aligned}\tag{4.5}$$

We have $\lim_{M\to\infty} v_M(x) = u^+(x)$ for a.e. $x\in\Omega$ and can apply Fatou's Lemma which results in

$$\begin{aligned}&\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx\\ &\le e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx\\ &\quad +\frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla(u^+)^{k+1}|^p dx + e_3\left(\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\\ &\quad +e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_5(kp+1)\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}. \end{aligned}\tag{4.6}$$

We have either

$$\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} \le 1 \quad\text{or}\quad \left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} \le \int_\Omega (u^+)^{(k+1)p}dx,$$

respectively, either

$$\left(\int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}} \le 1 \quad\text{or}\quad \left(\int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}} \le \int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma,$$

and finally, either

$$\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}} \le 1 \quad\text{or}\quad \left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}} \le \int_\Omega (u^+)^{(k+1)p}dx.$$

Using the calculation above to (4.6), we obtain

$$\left[\frac{kp+1}{(k+1)^p}-\frac{c_2\delta}{(k+1)^p}\right]\int_\Omega|\nabla(u^+)^{k+1}|^p dx$$
$$\leq (C(\delta)c_2+e_6(kp+1))\int_\Omega(u^+)^{(k+1)p}dx+e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma+e_8, \quad (4.7)$$

where the choice $\delta=\frac{kp+1}{2c_2}$ results in

$$\frac{kp+1}{2(k+1)^p}\int_\Omega|\nabla(u^+)^{k+1}|^p dx$$
$$\leq e_9(kp+1)\int_\Omega(u^+)^{(k+1)p}dx+e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma+e_8. \quad (4.8)$$

It should be pointed out that

$$C(\delta)=(\delta p)^{-\frac{q}{p}}\cdot\frac{1}{q}=\left(\frac{2c_2}{p}\right)^{\frac{q}{p}}\cdot\left(\frac{1}{kp+1}\right)^{\frac{q}{p}}\cdot\frac{1}{q}\leq e_{10}$$

with a positive constant e_{10} where we have set $e_9=e_{10}c_2+e_6$. Adding on both sides of (4.8) the positive integral $\frac{kp+1}{2(k+1)^p}\int_\Omega(u^+)^{(k+1)p}dx$ yields

$$\frac{kp+1}{2(k+1)^p}\left[\int_\Omega|\nabla(u^+)^{k+1}|^p dx+\int_\Omega(u^+)^{(k+1)p}dx\right]$$
$$\leq e_{11}(kp+1)\int_\Omega(u^+)^{(k+1)p}dx+e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma+e_8, \quad (4.9)$$

due to the fact that $\frac{kp+1}{2(k+1)^p}<kp+1$ for all $k\geq 0$. Next we want to estimate the boundary integral by an integral in the domain Ω. Using (3.1), (3.2) and Young's inequality yields

$$\begin{aligned}
&\int_{\partial\Omega}\left((u^+)^{k+1}\right)^p d\sigma\\
&=\|(u^+)^{k+1}\|^p_{L^p(\partial\Omega)}\\
&\leq c_4^p\|(u^+)^{k+1}\|^p_{B_{pp}^{\frac{1}{p}+\tilde\varepsilon}(\Omega)}\\
&\leq c_4^p c_5^p\|(u^+)^{k+1}\|^{\left(\frac{1}{p}+\tilde\varepsilon\right)p}_{W^{1,p}(\Omega)}\|(u^+)^{k+1}\|^{\left(1-\frac{1}{p}-\tilde\varepsilon\right)p}_{L^p(\Omega)}\\
&\leq c_4^p c_5^p\left(\delta'\|(u^+)^{k+1}\|^{(1+\tilde\varepsilon p)\tilde q}_{W^{1,p}(\Omega)}+C(\delta')\|(u^+)^{k+1}\|^{(p-1-\tilde\varepsilon p)\tilde q'}_{L^p(\Omega)}\right)\\
&=c_4^p c_5^p\left(\delta'\|(u^+)^{k+1}\|^p_{W^{1,p}(\Omega)}+C(\delta')\|(u^+)^{k+1}\|^p_{L^p(\Omega)}\right),
\end{aligned} \quad (4.10)$$

where $\tilde q=\frac{p}{1+\tilde\varepsilon p}$ and $\tilde q'=\frac{p}{p-1-\tilde\varepsilon p}$ satisfy $\frac{1}{\tilde q}+\frac{1}{\tilde q'}=1$ and δ' is a free parameter specified later. Note that the positive constant $C(\delta')$ depends on δ'. Applying

(4.10) to (4.9) shows

$$
\begin{aligned}
&\frac{kp+1}{2(k+1)^p}\left[\int_\Omega |\nabla (u^+)^{k+1}|^p dx + \int_\Omega (u^+)^{(k+1)p} dx\right]\\
&\quad \leq e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_8\\
&\quad \leq e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_{12}\delta'\|(u^+)^{k+1}\|^p_{W^{1,p}(\Omega)}\\
&\qquad + e_{12}C(\delta')\|(u^+)^{k+1}\|^p_{L^p(\Omega)} + e_8,
\end{aligned}
$$

where $e_{12} = e_7 c_4^p c_5^p$ is a positive constant. We take $\delta' = \frac{kp+1}{e_{12}4(k+1)^p}$ to get

$$
\begin{aligned}
&\left(\frac{kp+1}{2(k+1)^p} - e_{12}\frac{kp+1}{e_{12}4(k+1)^p}\right)\left[\int_\Omega |\nabla (u^+)^{k+1}|^p dx + \int_\Omega |(u^+)^{k+1}|^p dx\right]\\
&\quad \leq e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_{12}C(\delta')\|(u^+)^{k+1}\|^p_{L^p(\Omega)} + e_8\\
&\quad \leq e_{13}(kp+1+C(\delta'))\int_\Omega (u^+)^{(k+1)p}dx + e_8, \qquad (4.11)
\end{aligned}
$$

where it holds

$$
\begin{aligned}
kp+1+C(\delta') &= kp+1+\left(\frac{4e_{12}}{p}\right)^{\frac{q}{p}}\cdot\left(\frac{(k+1)^p}{kp+1}\right)^{\frac{q}{p}}\cdot\frac{1}{q}\\
&\leq e_{14}\left(kp+1+\left(\frac{(k+1)^{\frac{p}{p-1}}}{(kp+1)^{\frac{1}{p-1}}}\right)\right)\\
&\leq e_{15}(kp+1)^{\frac{p}{p-1}}.
\end{aligned}
$$

Applying the calculations above to (4.11) provides

$$
\begin{aligned}
&\frac{kp+1}{4(k+1)^p}\left[\int_\Omega |\nabla (u^+)^{k+1}|^p dx + \int_\Omega |(u^+)^{k+1}|^p dx\right]\\
&\quad \leq e_{16}(kp+1)^{\frac{p}{p-1}}\left[\int_\Omega (u^+)^{(k+1)p}dx + 1\right],
\end{aligned}
$$

equivalently

$$
\|(u^+)^{k+1}\|^p_{W^{1,p}(\Omega)} \leq (kp+1)^{\frac{1}{p-1}}(k+1)^p e_{17}\left[\int_\Omega (u^+)^{(k+1)p}dx + 1\right].
$$

By Sobolev's embedding theorem a positive constant e_{18} exists such that

$$
\|(u^+)^{k+1}\|_{L^{p^*}(\Omega)} \leq e_{18}\|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}, \qquad (4.12)
$$

where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = 2p$ if $p = N$. We get

$$
\begin{aligned}
&\|u^+\|_{L^{(k+1)p^*}(\Omega)} \\
&\quad = \|(u^+)^{k+1}\|_{L^{p^*}(\Omega)}^{\frac{1}{k+1}} \\
&\quad \le e_{18}^{\frac{1}{k+1}} \|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}^{\frac{1}{k+1}} \\
&\quad \le e_{18}^{\frac{1}{k+1}} \left((kp+1)^{\frac{1}{(p-1)p}}(k+1)\right)^{\frac{1}{k+1}} e_{17}^{\frac{1}{(k+1)p}} \left[\int_\Omega (u^+)^{(k+1)p})dx + 1\right]^{\frac{1}{(k+1)p}}.
\end{aligned}
$$

Since $((kp+1)^{\frac{1}{(p-1)p}}(k+1))^{\frac{1}{\sqrt{k+1}}} \ge 1$ and $\lim_{k\to\infty}((kp+1)^{\frac{1}{(p-1)p}}(k+1))^{\frac{1}{\sqrt{k+1}}} = 1$, there exists a constant $e_{19} > 1$ such that $((kp+1)^{\frac{1}{(p-1)p}}(k+1))^{\frac{1}{k+1}} \le e_{19}^{\frac{1}{\sqrt{k+1}}}$. We obtain

$$
\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{18}^{\frac{1}{k+1}} e_{19}^{\frac{1}{\sqrt{k+1}}} e_{17}^{\frac{1}{(k+1)p}} \left[\int_\Omega (u^+)^{(k+1)p})dx + 1\right]^{\frac{1}{(k+1)p}}. \tag{4.13}
$$

Now, we will use the bootstrap arguments similarly as in the proof of [8, Lemma 3.2] starting with $(k_1+1)p = p^*$ to get

$$
\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le c(k)
$$

for any finite number $k > 0$ which shows that $u^+ \in L^r(\Omega)$ for any $r \in (1, \infty)$. To prove the uniform estimate with respect to k we argue as follows. If there is a sequence $k_n \to \infty$ such that

$$
\int_\Omega (u^+)^{(k_n+1)p} dx \le 1,
$$

we have immediately

$$
\|u^+\|_{L^\infty(\Omega)} \le 1,
$$

(cf. the proof of [8, Lemma 3.2]). In the opposite case there exists $k_0 > 0$ such that

$$
\int_\Omega (u^+)^{(k+1)p} dx > 1
$$

for any $k \ge k_0$. Then we conclude from (4.13)

$$
\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{18}^{\frac{1}{k+1}} e_{19}^{\frac{1}{\sqrt{k+1}}} e_{20}^{\frac{1}{(k+1)p}} \|u^+\|_{L^{(k+1)p}}, \quad \text{for any } k \ge k_0, \tag{4.14}
$$

where $e_{20} = 2e_{17}$. Choosing $k := k_1$ such that $(k_1+1)p = (k_0+1)p^*$ yields

$$
\|u^+\|_{L^{(k_1+1)p^*}(\Omega)} \le e_{18}^{\frac{1}{k_1+1}} e_{19}^{\frac{1}{\sqrt{k_1+1}}} e_{20}^{\frac{1}{(k_1+1)p}} \|u^+\|_{L^{(k_1+1)p}(\Omega)}. \tag{4.15}
$$

Next, we can choose k_2 in (4.14) such that $(k_2+1)p = (k_1+1)p^*$ to get

$$
\begin{aligned}
\|u^+\|_{L^{(k_2+1)p^*}(\Omega)} &\le e_{18}^{\frac{1}{k_2+1}} e_{19}^{\frac{1}{\sqrt{k_2+1}}} e_{20}^{\frac{1}{(k_2+1)p}} \|u^+\|_{L^{(k_2+1)p}(\Omega)} \\
&= e_{18}^{\frac{1}{k_2+1}} e_{19}^{\frac{1}{\sqrt{k_2+1}}} e_{20}^{\frac{1}{(k_2+1)p}} \|u^+\|_{L^{(k_1+1)p^*}(\Omega)}.
\end{aligned} \tag{4.16}
$$

By induction we obtain

$$\begin{aligned}\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} &\le e_{18}^{\frac{1}{k_n+1}} e_{19}^{\frac{1}{\sqrt{k_n+1}}} e_{20}^{\frac{1}{(k_n+1)p}} \|u^+\|_{L^{(k_n+1)p}(\Omega)} \\ &= e_{18}^{\frac{1}{k_n+1}} e_{19}^{\frac{1}{\sqrt{k_n+1}}} e_{20}^{\frac{1}{(k_n+1)p}} \|u^+\|_{L^{(k_{n-1}+1)p^*}(\Omega)}, \end{aligned} \tag{4.17}$$

where the sequence (k_n) is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$ with $k_0 > 0$. One easily verifies that $k_n + 1 = (\frac{p^*}{p})^n$. Thus

$$\begin{aligned}&\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \\ &\quad = e_{18}^{\sum_{i=1}^n \frac{1}{k_i+1}} e_{19}^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} e_{20}^{\sum_{i=1}^n \frac{1}{(k_i+1)p}} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)}, \end{aligned} \tag{4.18}$$

with $r_n = (k_n+1)p^* \to \infty$ as $n \to \infty$. Since $\frac{1}{k_i+1} = (\frac{p}{p^*})^i$ and $\frac{p}{p^*} < 1$ there is a constant $e_{21} > 0$ such that

$$\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \le e_{21}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} < \infty. \tag{4.19}$$

Let us assume that $u^+ \notin L^\infty(\Omega)$. Then there exist $\eta > 0$ and a set A of positive measure in Ω such that $u^+(x) \ge e_{21}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta$ for $x \in A$. It follows that

$$\begin{aligned}\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} &\ge \left(\int_A |u^+(x)|^{(k_n+1)p^*}\right)^{\frac{1}{(k_n+1)p^*}} \\ &\ge \left(e_{21}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta\right) |A|^{\frac{1}{(k_n+1)p^*}}.\end{aligned}$$

Passing to the limes inferior in the inequality above yields

$$\liminf_{n\to\infty} \|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \ge e_{21}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta,$$

which is a contradiction to (4.19) and hence, $u^+ \in L^\infty(\Omega)$. In a similar way one shows that $u^- = \max\{-u, 0\} \in L^\infty(\Omega)$. This proves $u = u^+ - u^- \in L^\infty(\Omega)$. □

Remark 4.2. Note that the finiteness of the integrals

$$\int_\Omega |\nabla(u^+)^{k+1}|^p dx, \quad \int_\Omega |(u^+)^{k+1}|^p dx$$

is shown in the end of the proof of Theorem 4.1 by a suitable choice of the parameter k. This is a typical proceeding in the use of the Moser iteration (see, e.g. [8]).

Let us now suppose an additional condition to the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows.

(G4) g satisfies the condition

$$|g(x_1, s_1) - g(x_2, s_2)| \le L\left[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\right],$$

for all pairs $(x_1, s_1), (x_2, s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where M_0 is a positive constant and $\alpha \in (0, 1]$.

Theorem 4.3. *Let the conditions (A1)–(A3), (F1)–(F3) and (G1)–(G4) be satisfied. Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1). Then $u \in C^{1,\alpha}(\overline{\Omega})$.*

Proof. Theorem 4.1 implies $u \in L^\infty(\Omega)$. Moreover, we see at once that the assumptions (0.3a)–(0.3d) and (0.6) in [10] are satisfied which yields in view of [10, Theorem 2] the assertion. □

Example. Let $A = -\Delta_p, 1 < p < \infty$, be the negative $p-$Laplacian which is defined by

$$-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{where } \nabla u = (\partial u/\partial x_1, \dots, \partial u/\partial x_N). \tag{4.20}$$

The coefficients $a_i, i = 1, \dots, N$ are given by

$$a_i(x, s, \xi) = |\xi|^{p-2}\xi_i.$$

Thus, hypothesis (A1) is satisfied with $k_0 = 0$ and $c_0 = 1$. Hypothesis (A2) is a consequence of the inequalities from the vector-valued function $\xi \mapsto |\xi|^{p-2}\xi$ (see [2, Page 37]) and (A3) is satisfied with $c_1 = 1$ and $k_1 = 0$. Our equation in (1.1) gets the form

$$\begin{aligned} -\Delta_p u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= g(x, u) \quad \text{on } \partial\Omega, \end{aligned} \tag{4.21}$$

where $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of u with respect to $\partial\Omega$. Theorems 4.1 and 4.3 ensure under the assumptions (F1)–(F3) and (G1)–(G4) that every solution u of (4.21) satisfies $u \in L^\infty(\Omega)$ and $u \in C^{1,\alpha}(\overline{\Omega})$.

Acknowledgment

The author gratefully acknowledges the helpful suggestions of M. Wilke and R. Zacher during the preparation of the paper.

References

[1] Amann, H.: Linear and Quasilinear Parabolic Problems. Birkhäuser Boston Inc., Boston (1995)

[2] Carl, S., Le, V.K., Motreanu, D.: Nonsmooth variational problems and their inequalities. Springer Monographs in Mathematics. Springer, New York (2007)

[3] Dobrowolski, M.: Applied functional analysis. Functional Analysis, Sobolev Spaces and Elliptic Differential Equations. Springer, Berlin (2006)

[4] Dobrowolski, M.: private communication, 2008

[5] Drábek, P.: The least eigenvalues of nonhomogeneous degenerated quasilinear eigenvalue problems. Math. Bohem. **120**, 169–195 (1995)

[6] Drábek, P.: Nonlinear eigenvalue problem for p-Laplacian in $\mathbf{R}^N$. Math. Nachr. **173**, 131–139 (1995)

[7] Drábek, P., Hernández, J.: Existence and uniqueness of positive solutions for some quasilinear elliptic problems. Nonlinear Anal. **44**, 189–204 (2001)

[8] Drábek, P., Kufner, A., Nicolosi, F.: Quasilinear elliptic equations with degenerations and singularities. Walter de Gruyter & Co., Berlin (1997)

[9] Lê, A.: Eigenvalue problems for the p-Laplacian. Nonlinear Anal. **64**, 1057–1099 (2006)

[10] Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**, 1203–1219 (1988)

[11] Motreanu, D., Motreanu, V.V., Papageorgiou, N.S.: Nonlinear neumann problems near resonance. Indiana Univ. Math. J. **58**(3), 1257–1279 (2009)

[12] Runst, T., Sickel, W.: Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. Walter de Gruyter & Co., Berlin (1996)

[13] Sickel, W.: private communication, 2008

[14] Triebel, H.: Interpolation theory, function spaces, differential operators. VEB Deutscher Verlag der Wissenschaften, Berlin (1978)

[15] Triebel, H.: Theory of Function Spaces. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig (1983)

[16] Triebel, H.: Theory of function spaces II. Monographs in Mathematics, vol. 84. Birkhäuser Verlag, Basel (1992)

[17] Wei, J., Xu, X.: Uniqueness and a priori estimates for some nonlinear elliptic Neumann equations in $\mathbb{R}^3$. Pacific J. Math. **221**, 159–165 (2005)

[18] Zeidler, E.: Nonlinear Functional Analysis and its Applications. II/B. Springer, New York (1990)

[19] Zhu, M.: Uniqueness results through a priori estimates. I. A three-dimensional Neumann problem. J. Differ. Equ. **154**, 284–317 (1999)

P. Winkert
Department of Mathematics
Martin-Luther-University Halle-Wittenberg
06099 Halle, Germany
e-mail: patrick.winkert@mathematik.uni-halle.de

Received: 21 July 2009.
Accepted: 16 December 2009.