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Nonlinear systems with Hartman-type perturbations

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Abstract
We consider a nonlinear Lienard-type system driven by a nonlinear, nonhomogeneous differential operator and a maximal monotone map. On the Carathéodory perturbation we do not impose any global growth condition. Instead we employ a Hartman-type hypotheses. Using tools from fixed point theory and the theory of operators of monotone type, we prove two existence theorems.

Keywords Nonlinear nonhomogeneous differential operator · Maximal monotone map · Hartman condition · Leray–Schauder alternative principle · Lienard system

Mathematics Subject Classification 34A60 · 34B15

1 Introduction

In 1960, Hartman [4], see also Hartman [5], proved that the semilinear Dirichlet system

$$u''(t) = f(t, u(t)) \quad \text{on } T = [0, b], \quad u(0) = u(b) = 0$$

with $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ being continuous, admits a solution provided that there exists $M > 0$ such that

$$(f(t, x), x)_{\mathbb{R}^N} \geq 0 \quad \text{for all } t \in T \text{ and for all } x \in \mathbb{R}^N \text{ with } |x| = M.$$

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Later, Knobloch [6] extended the result to semilinear periodic systems under the assumption that the vector field $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz. More recently, Mawhin [8] extended the results of Hartman and Knobloch to nonlinear systems driven by the vector p-Laplacian and having a continuous vector field $f : T \times \mathbb{R}^N \to \mathbb{R}^N$.

In this paper we go well beyond the aforementioned works and deal with the following nonlinear system:

$$
\begin{aligned}
&a\left(u'(t)\right)' + \frac{d}{dt}\nabla G(u(t)) \in A(u(t)) + f(t, u(t)) \quad \text{for a.a. } t \in T = [0, b],\\
&\qquad\qquad u \in \mathrm{BC},
\end{aligned}
\tag{1.1}
$$

where we mean by $u \in \mathrm{BC}$ that u satisfies one of the following boundary conditions

- Dirichlet condition: $u(0) = u(b) = 0$;
- Neumann condition: $u'(0) = u'(b) = 0$;
- Periodic condition: $u(0) = u(b)$, $u'(0) = u'(b)$.

We will do the proof for the periodic problem and the same reasoning, in fact in a simpler form, applies also to the other two boundary conditions.

In problem (1.1), the mapping $a : \mathbb{R}^N \to \mathbb{R}^N$ is a suitable homeomorphism, in general nonhomogeneous, which includes many differential operators of interest as special cases such as the vector p-Laplacian. For G we suppose $G \in C^2(\mathbb{R}^N, \mathbb{R})$ and on the right-hand side of (1.1), $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map and $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory perturbation, that is, $t \to f(t, x)$ is measurable for all $x \in \mathbb{R}^N$ and $x \to f(t, x)$ is continuous for a.a. $t \in T$. We do not assume that the domain of A is all of $\mathbb{R}^N$ and this incorporates in our framework systems with unilateral constraints, namely differential variational inequalities. Moreover, we do not impose any global growth condition on the perturbation term $f(t, \cdot)$. Instead we employ the Hartman-type condition mentioned in the beginning of the paper. The particular form of (1.1) classifies the problem as a nonlinear Lienard system, see Hartman [5, p. 179].

Our approach uses tools from fixed point theory and from the theory of nonlinear operators of monotone type.

2 Preliminaries and hypotheses

Let X be a reflexive Banach space, let X^* be its topological dual and denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). We say that a map $A : X \to 2^{X^*}$ is monotone if

$$
\left\langle u^* - x^*, u - x \right\rangle \geq 0 \quad \text{for all } (u, u^*), (x, x^*) \in \mathrm{Gr}\, A,
$$

where

$$
\mathrm{Gr}\, A = \left\{ (v, v^*) \in X \times X^* : v^* \in A(v) \right\}
$$

denotes the graph of A. If A satisfies

$$\langle u^* - x^*, u - x \rangle > 0 \quad \text{for all } (u, u^*), (x, x^*) \in \operatorname{Gr} A \text{ with } u \neq x,$$

then we say that A is strictly monotone. Finally we say that $A : X \to 2^{X^*}$ is maximal monotone if

$$\langle u^* - x^*, u - x \rangle \geq 0 \quad \text{for all } (u, u^*) \in \operatorname{Gr} A \quad \text{implies} \quad (x, x^*) \in \operatorname{Gr} A.$$

This means that $\operatorname{Gr} A$ is maximal with respect to inclusion among the graphs of all monotone maps. By $D(A)$ we denote the domain of A, that is,

$$D(A) = \{u \in X : A(u) \neq \emptyset\}.$$

For a maximal monotone map A we have that $\operatorname{Gr} A$ is sequentially closed in $X_{\mathrm{w}} \times X^*$ and in $X \times X^*_{\mathrm{w}}$.

Now, let H be a Hilbert space. We identify H with its dual by the Fréchet–Riesz theorem, that is, $H = H^*$. Let $A : H \to 2^H$ be a maximal monotone map. For $\lambda > 0$ we define the following single-valued maps

$$\text{Resolvent of } A : \quad J_\lambda = (I + \lambda A)^{-1},$$
$$\text{Yosida approximation of } A : \quad A_\lambda = \frac{1}{\lambda}[I - J_\lambda].$$

The next proposition summarizes the main properties of these two operators.

Proposition 2.1 *If $A : H \to 2^H$ is a maximal monotone map and $\lambda > 0$, then the following hold:*

(a) $J_\lambda : H \to H$ *is nonexpansive, that is* $\|J_\lambda(u) - J_\lambda(x)\| \leq \|u - x\|$ *for all* $u, x \in H$;
(b) $A_\lambda(u) \in A(J_\lambda(u))$ *for all* $u \in H$;
(c) A_λ *is monotone and* $\|A_\lambda(u) - A(x)\| \leq \frac{1}{\lambda}\|u - x\|$ *for all* $u, x \in H$;
(d) $\|A_\lambda(u)\| \leq \|A^0(u)\| = \min\{\|u^*\| : u^* \in A(u)\}$ *and* $A_\lambda(u) \to A^0(u)$ *as* $\lambda \to 0^+$ *for all* $u \in D(A)$;
(e) $\overline{D(A)}$ *is convex and* $J_\lambda(u) \to \operatorname{proj}\left(u; \overline{D(A)}\right)$ *for all* $u \in H$.

Remark 2.2 The maximal monotonicity of A implies that $A(u) \subseteq H$ is nonempty, closed and convex for all $u \in D(A)$. Therefore, the minimal norm element $A^0(u)$ exists. Moreover, $\overline{D(A)}$ is convex and so the metric projection $\operatorname{proj}(\cdot, \overline{D(A)})$ is well-defined. For more about maps of monotone type we refer to Papageorgiou–Winkert [9].

Suppose that V, Z are Banach spaces and let $K : V \to Z$. We introduce the following two notions:

- We say that K is completely continuous if

$$v_n \xrightarrow{\mathrm{w}} v \quad \text{in } V \quad \text{implies} \quad K(v_n) \to K(v) \quad \text{in } Z.$$

- We say that K is compact if it is continuous and maps bounded sets in V to relatively compact sets in Z.

From the fixed point theory, we will use the Leray–Schauder Alternative Principle which says the following.

Theorem 2.3 *If V is a Banach space, $K : V \to V$ is a compact map and*

$$S = \{v \in V : v = \mu K(v) \text{ for some } 0 < \mu < 1\},$$

then one of the following two statements is true:

(a) *S is unbounded;*
(b) *K has a fixed point.*

By $\rho_M : \mathbb{R}^N \to \mathbb{R}^N$ with $M > 0$ with denote the map

$$\rho_M(u) = \begin{cases} u & \text{if } |u| \le M, \\ \frac{Mu}{|u|} & \text{if } M < |u|, \end{cases}$$

for all $u \in \mathbb{R}^N$, where we denote by $|u|$ the Euclidean norm of u for every $u \in \mathbb{R}^N$. It is easy to see that the map ρ_m is nonexpansive.

For notational simplicity, we will write $W^{1,p}$ with $1 < p < \infty$ for the space $W^{1,p}((0,b), \mathbb{R}^N)$ and by $\|\cdot\|$ we will denote the norm of $W^{1,p}$ defined by

$$\|u\| = \left(\|u\|_p^p + \|u'\|_p^p\right)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}.$$

Given a function $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ we denote by N_f the Nemytskij operator corresponding to f defined by

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \quad \text{for all } u \in W^{1,p}.$$

Now we introduce the hypotheses on the data of (1.1).

H(a): $a : \mathbb{R}^N \to \mathbb{R}^N$ is a strictly monotone, continuous map such that $a(0) = 0$,

$$a(y) = c(|y|)y \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}$$

with a continuous function $c : (0, +\infty) \to (0, +\infty)$ and there exist $c_0 > 0$ and $1 < p < \infty$ such that

$$c_0|y|^p \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N.$$

Remark 2.4 Evidently, a is maximal monotone. Furthermore, a is a homeomorphism onto $\mathbb{R}^N$ and $|a^{-1}(y)| \to +\infty$ as $|y| \to +\infty$. We stress that no growth condition is imposed on a.

Example 2.5 The following maps satisfy hypotheses H(a):

- $a(y) = |y|^{p-2}y$ with $1 < p < \infty$,
- $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$,
- $a(y) = \left[1 + |y|^2\right]^{\frac{p-2}{2}} y$ with $1 < p < \infty$,
- $a(y) = \left[ce^{|y|^p} - 1\right]|y|^{p-2}y$ with $1 < p < \infty$ and $c > 1$,

for all $y \in \mathbb{R}^N$. The first map corresponds to the vector p-Laplacian and the second one to the vector (p, q)-Laplacian.

The assumptions on G read as follows:

H(G): $G \in C^2(\mathbb{R}^N, \mathbb{R})$ and $\nabla G(x) = g_0(|x|)x$ for all $x \in \mathbb{R}^N$ with $g_0(r) > 0$ for all $r > 0$.

Remark 2.6 As mentioned before, we do not assume any global growth condition on the function G.

Example 2.7 The following maps fulfill H(G):

- $G(x) = \frac{1}{r}|x|^r$ with $2 \leq r < \infty$,
- $G(x) = \frac{1}{r}|x|^r + \frac{1}{q}|x|^q$ with $2 \leq q < r < \infty$,
- $G(x) = \frac{1}{2}\left[e^{|x|^2} - 1\right]$,

for all $x \in \mathbb{R}^N$.

Finally, we can state our assumptions on $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ and $f : T \times \mathbb{R}^N \to \mathbb{R}^N$.

H(A): $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map with $0 \in A(0)$;

H(f): $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

(i) for every $\eta > 0$ there exists $a_\eta \in L^2(T)_+$ such that

$$|f(t, x)| \leq a_\eta(t) \quad \text{for a.a. } t \in T \text{ and for all } |x| \leq \eta;$$

(ii) there exists $M > 0$ such that

$$(f(t, x), x)_{\mathbb{R}^N} \geq 0$$

for a.a. $t \in T$ and for all $x \in \mathbb{R}^N$ with $|x| = M$.

3 Existence of solutions

For $h \in L^1(T, \mathbb{R}^N)$ we consider the following system

$$\begin{aligned} -a\left(u'(t)\right)' + |u(t)|^{p-2}u'(t) = h(t) \qquad &\text{for a.a. } t \in T, \\ u(0) = u(b),\ u'(0) = u'(b). \end{aligned} \tag{3.1}$$

Proposition 3.1 *If hypotheses H(a) hold, then problem (3.1) has a unique solution $K(h) \in C^1(T, \mathbb{R}^N)$ for every $h \in L^1(T, \mathbb{R}^N)$.*

Proof Note that

$$\int_0^b \left[h(t) - |u(t)|^{p-2} u(t) \right] dt = 0.$$

The existence of a solution $K(h) \in C^1(T, \mathbb{R}^N)$ follows from Theorem 5.3 of Manásevich–Mawhin [7]. The uniqueness of this solution is a consequence of the strict monotonicity of the maps

$$\mathbb{R}^N \ni y \to a(y) \quad \text{and} \quad \mathbb{R}^N \ni x \to |x|^{p-2} x.$$

□

Remark 3.2 The above proposition is stated in a little more general form than we will need it here. Indeed, it is enough to consider $h \in L^2(T, \mathbb{R}^N)$, see hypothesis H(f)(i). However, when $D(A) = \mathbb{R}^N$, then we can have $a_\eta \in L^1(T)_+$ in hypothesis H(f)(i) and so we use Proposition 3.1. For the Dirichlet problem, on account of the Poincaré inequality, we consider instead of (3.1) the following problem

$$\begin{aligned} -a \left(u'(t) \right)' &= h(t) \qquad \text{for a.a. } t \in T, \\ u(0) &= u(b) = 0. \end{aligned}$$

Then, the existence and uniqueness of a solution $K(h) \in C^1(T, \mathbb{R}^N)$ follows from Theorem 5.1 of Manásevich–Mawhin [7].

Now we can define the solution map $K : L^1(T, \mathbb{R}^N) \to C^1(T, \mathbb{R}^N)$ and obtain the following property of this map.

Proposition 3.3 *If hypotheses H(a) hold, then K is completely continuous.*

Proof Let $h_n \xrightarrow{w} h$ in $L^1(T, \mathbb{R}^N)$ and set $u_n = K(h_n)$ for all $n \in \mathbb{N}$. We have for $n \in \mathbb{N}$

$$\begin{aligned} -a \left(u_n'(t) \right)' + |u_n(t)|^{p-2} u_n(t) &= h_n(t) \qquad \text{for a.a. } t \in T, \\ u_n(0) = u_n(b), \ u_n'(0) &= u_n'(b). \end{aligned} \tag{3.2}$$

We take the inner product with $u_n(t)$, integrate over $T = [0, b]$ and perform integration by parts. This leads to

$$\int_0^b (a(u_n'), u_n')_{\mathbb{R}^N} dt + \|u_n\|_p^p \leq c_1 \|u_n\| \quad \text{for some } c_1 > 0 \text{ and for all } n \in \mathbb{N}.$$

Taking hypotheses H(a) into account gives

$$c_0 \|u_n'\|_p^p + \|u_n\|_p^p \leq c_1 \|u_n\| \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}$ is bounded and since $W^{1,p} \hookrightarrow C(T, \mathbb{R}^N)$ is compactly embedded, we conclude that

$$\{u_n\}_{n\geq 1} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.} \tag{3.3}$$

From (3.2) we have

$$a(u_n'(t)) = a(u_n'(0)) + \int_0^t \left[h_n(s) - |u_n(s)|^{p-2}u_n(s)\right] ds \tag{3.4}$$

for all $t \in T$ and for all $n \in \mathbb{N}$. This gives

$$u_n'(t) = a^{-1}\left[a(u_n'(0)) + \int_0^t \left[h_n(s) - |u_n(s)|^{p-2}u_n(s)\right] ds\right]$$

for all $t \in T$ and for all $n \in \mathbb{N}$. If

$$k_n(t) = \int_0^t \left[h_n(s) - |u_n(s)|^{p-2}u_n(s)\right] ds$$

for $n \in \mathbb{N}$, then $\{k_n\}_{n\in\mathbb{N}} \subseteq C(T, \mathbb{R}^N)$ is bounded. Moreover, note that $\int_0^t u_n'(t)dt = 0$ for $n \in \mathbb{N}$. Therefore, Lemma 3.1 of Manásevich–Mawhin [7] implies that

$$\{a(u_n(0))\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^N \text{ is bounded.}$$

Then, from (3.4) and the Arzela–Ascoli theorem, we infer that

$$\{a(u_n'(\cdot))\}_{n\in\mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.} \tag{3.5}$$

Let $\hat{a}^{-1} : C(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ be defined by

$$\hat{a}^{-1}(u)(\cdot) = a^{-1}(u(\cdot)) \quad \text{for all } u \in C(T, \mathbb{R}^N).$$

Evidently, $\hat{a}^{-1}$ is continuous and bounded, that is, it maps bounded sets to bounded sets. Hence, from (3.5) we have

$$\{u_n'\}_{n\in\mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.} \tag{3.6}$$

From (3.3) and (3.6) it follows that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq C^1(T, \mathbb{R}^N) \text{ is relatively compact.}$$

We may assume, at least for a subsequence, that

$$u_n \to u \quad \text{in } C^1(T, \mathbb{R}^N). \tag{3.7}$$

We have

$$\int_0^b (a(u_n'), v')_{\mathbb{R}^N} dt + \int_0^b |u_n|^{p-2}(u_n, v)_{\mathbb{R}^N} dt = \int_0^b (h_n, v)_{\mathbb{R}^N} dt \tag{3.8}$$

for all $v \in W^{1,p}$ and for all $n \in \mathbb{N}$. From (3.7) and the continuity of a, we obtain

$$|a(u_n'(t))| \le c_2$$

for some $c_2 > 0$, for all $t \in T$ and for all $n \in \mathbb{N}$. So, if we pass to the limit in (3.8) as $n \to \infty$, then one has

$$\int_0^b (a(u'), v')_{\mathbb{R}^N} dt + \int_0^b |u|^{p-2}(u, v)_{\mathbb{R}^N} dt = \int_0^b (h, v)_{\mathbb{R}^N} dt$$

for all $v \in W^{1,p}$. Hence, $u = K(h)$.

Therefore, we obtain for the original sequence that

$$u_n = K(h_n) \to K(h) = u \quad \text{in } C^1(T, \mathbb{R}^N),$$

which shows that $K : L^1(T, \mathbb{R}^N) \to C^1(T, \mathbb{R}^N)$ is completely continuous. □

Remark 3.4 In particular, we obtain that $K : L^2(T, \mathbb{R}^N) \to C^1(T, \mathbb{R}^N)$ is completely continuous and then, due to the reflexivity of $L^2(T, \mathbb{R}^N)$, we have that K is compact, see Papageorgiou–Winkert [9, Proposition 3.7.7].

For every $\lambda > 0$, let $\hat{A}_\lambda : W^{1,p} \to L^2(T, \mathbb{R}^N)$ be defined by $\hat{A}_\lambda(u)(\cdot) = A_\lambda(u(\cdot))$. In fact, $\hat{A}_\lambda$ is $L^\infty(T, \mathbb{R}^N)$-valued. Then, let $N_\lambda : W^{1,p} \to L^2(T, \mathbb{R}^N)$ be defined by

$$N_\lambda(u) = -\hat{A}_\lambda(u) - N_f(\rho_M(u)) + |\rho_M(u)|^{p-2}\rho_M(u) + \nabla G(\rho_M(u)).$$

The following proposition is an immediate consequence of the properties of A_λ, see Proposition 2.1, and of the hypotheses H(G) and H(f).

Proposition 3.5 *If hypotheses H(A), H(G) and H(f) hold, then $N_\lambda : W^{1,p} \to L^2(T, \mathbb{R}^N)$ is continuous.*

From Propositions 3.3 and 3.5 we easily conclude that the map $K \circ N_\lambda : W^{1,p} \to W^{1,p}$ is compact. We define

$$S_\lambda = \left\{ u \in W^{1,p} : u = \mu(K \circ N_\lambda)(u), \ 0 < \mu < 1 \right\}.$$

Proposition 3.6 *If hypotheses H(a), H(A), H(G), H(f) hold and $\lambda > 0$, then $S_\lambda \subseteq W^{1,p}$ is bounded.*

Proof Let $u \in S_\lambda$. Then $\frac{1}{\mu}u = K(N_\lambda(u))$ and so

$$\begin{aligned} &-a\left(\frac{1}{\mu}u'\right)' + \frac{1}{\mu^{p-1}}|u|^{p-2}u \\ &\quad = -\hat{A}_\lambda(u) - N_f(\rho_M(u)) + |\rho_M(u)|^{p-2}\rho_M(u) + \frac{d}{dt}\nabla G(\rho_M(u)) \end{aligned} \tag{3.9}$$

with $u(0) = u(b)$ and $u'(0) = u'(b)$.

Claim: $|u(t)| \le M$ for all $t \in T$

Let $r(t) = \frac{1}{2}|u(t)|^2$ for all $t \in T$. Then we can find $t_0 \in T$ such that $r(t_0) = \max_T r$. Arguing by contradiction, suppose that

$$r(t_0) > \frac{1}{2}M^2.$$

First we assume that $t_0 \in (0, b)$. Then

$$r'(t_0) = (u'(t_0), u(t_0))_{\mathbb{R}^N} = 0. \tag{3.10}$$

Let $t_1 \in [0, t_0)$ be such that $|u(t_1)| = M$ and $|u(t)| > M$ for all $(t_1, t_0]$. Then

$$\begin{aligned} &-a\left(\frac{1}{\mu}u'(t)\right)' + \frac{1}{\mu^{p-1}}|u(t)|^{p-2}u(t) \\ &\quad = -\hat{A}_\lambda(u(t)) - f(t, \rho_M(u(t))) + |\rho_M(u(t))|^{p-2}\rho_M(u(t)) + \frac{d}{dt}\nabla G(\rho_M(u(t))) \end{aligned}$$

for a.a. $t \in T$. This implies

$$\begin{aligned} &-\frac{d}{dt}\left(a\left(\frac{1}{\mu}u'(t)\right), u(t)\right)_{\mathbb{R}^N} + \left(a\left(\frac{1}{\mu}u'(t)\right), u'(t)\right)_{\mathbb{R}^N} + \frac{1}{\mu^{p-1}}|u(t)|^p \\ &\quad = -(A_\lambda(u(t)), u(t))_{\mathbb{R}^N} - \frac{|u(t)|}{M}(f(t, \rho_M(u(t))), \rho_M(u(t)))_{\mathbb{R}^N} \\ &\qquad + |u(t)|M^{p-1} + \left(\frac{d}{dt}\nabla G(\rho_M(u(t))), u(t)\right)_{\mathbb{R}^N} \end{aligned} \tag{3.11}$$

for a.a. $t \in [t_1, t_0]$. Since A_λ is maximal monotone, see Proposition 2.1, and $A_\lambda(0) = 0$, see hypotheses H(a), we have

$$-(A_\lambda(u(t)), u(t))_{\mathbb{R}^N} \le 0 \quad \text{for all } t \in T. \tag{3.12}$$

Furthermore, taking hypothesis H(f)(ii) into account, we obtain

$$-\frac{|u(t)|}{M}(f(t, \rho_M(u(t))), \rho_M(u(t)))_{\mathbb{R}^N} \le 0 \quad \text{for all } t \in [t_1, t_0]. \tag{3.13}$$

Finally, applying hypotheses H(G), we have

$$\begin{aligned}
&\left(\frac{d}{dt}\nabla G(\rho_M(u(t))), u(t)\right)_{\mathbb{R}^N} \\
&= \frac{|u(t)|}{M}\left(\frac{d}{dt}\nabla G(\rho_M(u(t))), \rho_M(u(t))\right)_{\mathbb{R}^N} \\
&= \frac{|u(t)|}{M}\left[\frac{d}{dt}\left(\nabla G(\rho_M(u(t))), \rho_M(u(t))\right)_{\mathbb{R}^N}\right. \\
&\quad \left. - \left(\nabla G(\rho_M(u(t))), \frac{d}{dt}\rho_M(u(t))\right)_{\mathbb{R}^N}\right] \\
&= \frac{|u(t)|}{dt}\left[\frac{d}{dt}\left(g_0(M)M^2\right) - g_0(M)\frac{d}{dt}|\rho_M(u(t))|^2\right] = 0
\end{aligned} \tag{3.14}$$

for all $t \in [t_1, t_0]$. We return to (3.11) and apply (3.12), (3.13), (3.14) and hypotheses H(a). This gives

$$|u(t)|\left[\frac{1}{\mu^{p-1}}|u(t)|^{p-1} - M^{p-1}\right] \le \frac{d}{dt}\left(a\left(\frac{1}{\mu}u'(t)\right), u(t)\right)_{\mathbb{R}^N}$$

for a.a. $t \in (t_1, t_0]$ and so, since $0 < \mu < 1$,

$$0 < \frac{d}{dt}\left(a\left(\frac{1}{\mu}u'(t)\right), u(t)\right)_{\mathbb{R}^N} \quad \text{for a.a. } t \in (t_1, t_0].$$

Therefore, the function

$$t \to \left(a\left(\frac{1}{\mu}u'(t)\right), u(t)\right)_{\mathbb{R}^N}$$

is strictly increasing on $(t_1, t_0]$. Hence, we have

$$\left(a\left(\frac{1}{\mu}u'(t)\right), u(t)\right)_{\mathbb{R}^N} < \left(a\left(\frac{1}{\mu}u'(t_0)\right), u(t_0)\right)_{\mathbb{R}^N} \quad \text{for all } t \in (t_1, t_0).$$

Based on hypotheses H(a) and (3.10) we obtain

$$c\left(\frac{1}{\mu}|u'(t)|\right)\left(u'(t), u(t)\right)_{\mathbb{R}^N} < c\left(\frac{1}{\mu}|u'(t_0)|\right)\left(u'(t_0), u(t_0)\right)_{\mathbb{R}^N} = 0.$$

Thus, $r'(t) < 0$ for all $t \in (t_1, t_0)$.

Finally we have

$$M^2 < r(t_0) < r(t_1) = M^2,$$

a contradiction.

If $t_0 = 0$ or $t_0 = b$, then $r(0) = r(b)$ and $r'(0) \leq 0 \leq r'(b)$. But

$$r'(t) = \left(u'(t), u(t)\right)_{\mathbb{R}^N} \quad \text{for all } t \in T,$$

which implies $r'(0) = r'(b) = 0$ and so the previous argument applies. This proves the Claim.

Next we act on (3.9) with u, perform integration by parts and use hypotheses H(a), H(G), H(f)(i) and the Claim. This gives

$$\frac{1}{\mu^{p-1}}\left[c_0 \left\|u'\right\|_p^p + \|u\|_p^p\right] \leq c_3 \quad \text{for some } c_3 > 0 \text{ and for all } u \in S.$$

Recall that $0 < \mu < 1$, we see that $S \subseteq W^{1,p}$ is bounded. □

For $\lambda > 0$ we consider the following approximation to problem (1.1)

$$\begin{aligned} &a\left(u'(t)\right)' + \frac{d}{dt}\nabla G(u(t)) = A_\lambda(u(t)) + f(t, u(t)) \qquad \text{for a.a. } t \in T = [0, b],\\ &\quad u(0) = u(b),\ u'(0) = u'(b). \end{aligned} \tag{3.15}$$

Proposition 3.7 *If hypotheses H(a), H(G), H(A), H(f) hold and let* $\lambda > 0$, *then problem* (3.15) *has a unique solution* $\hat{u}_\lambda \in C^1(T, \mathbb{R}^N)$.

Proof The compactness of $K \circ N_\lambda : W^{1,p} \to W^{1,p}$ and Proposition 3.6 permit the use of the Leray–Schauder Alternative Principle stated as Theorem 2.3. So, there exists $\hat{u}_\lambda \in W^{1,p}$ such that

$$\hat{u} = (K \circ N_\lambda)\left(\hat{u}_\lambda\right).$$

This gives

$$\hat{u}_\lambda \in C^1(T, \mathbb{R}^N) \quad \text{and} \quad \left|\hat{u}_\lambda(t)\right| \leq M \quad \text{for all } t \in T,$$

see the proof of Proposition 3.6. Then $\rho_M(\hat{u}_\lambda(t)) = \hat{u}_\lambda(t)$ and so we conclude that $\hat{u}_\lambda \in C^1(T, \mathbb{R}^N)$ is a solution of (3.15), see (3.9) with $\mu = 1$. □

Let $\mathfrak{a} : L^2(T, \mathbb{R}^N) \to 2^{L^2(T, \mathbb{R}^N)}$ be defined by

$$\mathfrak{a}(u) = \left\{\vartheta \in L^2(T, \mathbb{R}^N) : \vartheta(t) \in A(u(t)) \text{ for a.a. } t \in T\right\}.$$

Since $0 \in A(0)$ we see that $D(\mathfrak{a}) \neq \emptyset$. From Brézis [2, p. 21] we have the following result.

Proposition 3.8 *If hypotheses H(A) hold, then* $\mathfrak{a}$ *is maximal monotone.*

Now we are ready to produce a solution for problem (1.1).

Theorem 3.9 *If hypotheses H(a), H(G), H(A), H(f) hold, then problem* (1.1) *has a solution* $\hat{u} \in C^1(T, \mathbb{R}^N)$.

Proof Let $\lambda_n \to 0^+$ and let $\hat{u}_n = \hat{u}_{\lambda_n} \in C^1(T, \mathbb{R}^N)$ for $n \in \mathbb{N}$ be a solution of (3.15) based on Proposition 3.7. From the proof of Proposition 3.6, see the Claim in that proof, we have

$$\left|\hat{u}_n(t)\right| \leq M \quad \text{for all } t \in T \text{ and for all } n \in \mathbb{N}. \tag{3.16}$$

From (3.15) it follows that

$$\int_0^b \left(a\left(\hat{a}_n'\right), \hat{u}_n'\right)_{\mathbb{R}^N} dt \leq \int_0^b \left|f\left(t, \hat{u}_n\right)\right| M dt + \int_0^b \left|\frac{d}{dt}\nabla G\left(\hat{u}_n\right)\right| M dt,$$

where we recall that A_{λ_n} is monotone, $A_{\lambda_n}(0) = 0$ and see (3.16). Applying hypotheses H(a), H(G) and H(f)(i) leads to

$$c_0 \left\|\hat{u}_n'\right\|_p^p \leq c_3 \quad \text{for some } c_3 > 0 \text{ and for all } n \in \mathbb{N}.$$

Therefore, the sequence $\{\hat{u}_n'\}_{n\in\mathbb{N}} \subseteq L^p(T, \mathbb{R}^N)$ is bounded and so it is $\{\hat{u}_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}$, see (3.16). So, by passing to a subsequence if necessary, we can say that

$$\hat{u}_n \overset{w}{\to} \hat{u} \quad \text{in } W^{1,p} \quad \text{and} \quad \hat{u}_n \to \hat{u} \quad \text{in } C(T, \mathbb{R}^N).$$

Now we take the inner product with $A_{\lambda_n}(\hat{u}_n(t))$ in (3.15) and integrate over $T = [0, b]$. After integration by parts and by applying hypotheses H(G), H(f)(i) and (3.16), we obtain

$$\int_0^b \left(a\left(\hat{u}_n'\right), \frac{d}{dt} A_{\lambda_n}(u_n)\right)_{\mathbb{R}^N} dt + \left\|A_{\lambda_n}(u_n)\right\|_2^2 \leq c_4 \left\|A_{\lambda_n}(u_n)\right\|_2 \tag{3.17}$$

for some $c_4 > 0$ and for all $n \in \mathbb{N}$.

The map $x \to A_{\lambda_n}(x)$ for $n \in \mathbb{N}$ is Lipschitz continuous from $\mathbb{R}^N$ into $\mathbb{R}^N$. So, by the Rademacher theorem, see Evans–Gariepy [3, p. 81], we know that A_{λ_n} is differentiable at all $x \in \mathbb{R}^N \setminus D_n$ with $|D_n|_N = 0$, where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. Then, since A_{λ_n} is monotone, we have for all $x \in \mathbb{R}^N \setminus D_n$ and for every $h \in \mathbb{R}^N$,

$$\left(\frac{A_{\lambda_n}(x + \tau h) - A_{\lambda_n}(x)}{\tau}, h\right)_{\mathbb{R}^N} \geq 0.$$

This implies

$$\left(A_{\lambda_n}'(x)h, h\right)_{\mathbb{R}^N} \geq 0. \tag{3.18}$$

Then, from the chain rule for Sobolev functions, see Papageorgiou–Winkert [9, Theorem 4.5.18], we have

$$\frac{d}{dt}A_{\lambda_n}\left(\hat{u}_n(t)\right) = A'_{\lambda_n}\left(\hat{u}_n(t)\right)\hat{u}'_n(t) \quad \text{for a.a. } t \in T. \tag{3.19}$$

Applying (3.19), hypotheses H(a) and (3.18) gives

$$\begin{aligned}
&\int_0^b \left(a\left(\hat{u}'_n(t)\right), \frac{d}{dt}A_{\lambda_n}\left(\hat{u}_n\right)\right)_{\mathbb{R}^N} dt \\
&= \int_0^b \left(a\left(\hat{u}'_n(t)\right), A'_{\lambda_n}\left(\hat{u}_n\right)\hat{u}'_n\right)_{\mathbb{R}^N} dt \\
&= \int_0^b c\left(\left|\hat{u}'_n\right|\right)\left(\hat{u}'_n, \hat{A}_{\lambda_n}(u_n)\hat{u}_n\right)_{\mathbb{R}^N} dt \geq 0.
\end{aligned}$$

Returning to (3.17) and using (3.19) we obtain

$$\left\| A_{\lambda_n}(u_n)\right\|_2^2 \leq c_4 \left\| A_{\lambda_n}(u_n)\right\|_2 \quad \text{for all } n \in \mathbb{N},$$

which shows that

$$\left\{\hat{A}_{\lambda_n}(u_n)\right\}_{n\in\mathbb{N}} = \left\{A_{\lambda_n}(u_n(\cdot))\right\}_{n\in\mathbb{N}} \subseteq L^2(T, \mathbb{R}^N) \text{ is bounded.}$$

So, we may assume that

$$\hat{A}_{\lambda_n}\left(\hat{u}_n\right) \xrightarrow{\text{w}} y \quad \text{in } L^2(T, \mathbb{R}^N). \tag{3.20}$$

From (3.15) we have

$$u'_n(t) = a^{-1}\left(a(u'_n(0)) + \int_0^t \left[A_{\lambda_n}\left(\hat{u}_n(s)\right) + f\left(s, \hat{u}_n(s)\right) - \frac{d}{dt}\nabla G\left(\hat{u}_n(s)\right)\right] ds\right) \tag{3.21}$$

for all $n \in \mathbb{N}$. We set

$$g_n(t) = \int_0^t \left[A_{\lambda_n}\left(\hat{u}_n(s)\right) + f\left(s, \hat{u}_n(s)\right) - \frac{d}{dt}\nabla G\left(\hat{u}_n(s)\right)\right] ds$$

for all $t \in T$ and for all $n \in \mathbb{N}$. The Arzela–Ascoli theorem implies that

$$\{g_n\}_{n\in\mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.}$$

Therefore, invoking Lemma 3.1 of Manásevich–Mawhin [7], we infer that

$$\{a(u'_n(0))\}_{n\geq 1} \subseteq \mathbb{R}^N \text{ is relatively compact.}$$

Recall that the map $\hat{a}^{-1} : C(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ defined by $\hat{a}^{-1}(u)(\cdot) = a^{-1}(u(\cdot))$ is continuous. Thus, from (3.21) it follows that

$$\{\hat{u}_n'\}_{n\in\mathbb{N}} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact}$$

and because of the compact embedding $W^{1,p} \hookrightarrow C(T, \mathbb{R}^N)$,

$$\{\hat{u}_n\}_{n\in\mathbb{N}} \subseteq C^1(T, \mathbb{R}^N) \text{ is relatively compact.}$$

So, we have

$$\hat{u}_n \to \hat{u} \quad \text{in } C^1(T, \mathbb{R}^N). \tag{3.22}$$

In the limit as $n \to \infty$, we obtain

$$\begin{aligned} &-\int_0^b \left(a\left(\hat{u}'\right), v'\right)_{\mathbb{R}^N} dt + \int_0^b \left(\frac{d}{dt}\nabla G\left(\hat{u}\right), v\right)_{\mathbb{R}^N} dt \\ &= \int_0^b (y, v)_{\mathbb{R}^N} dt + \int_0^b \left(f\left(t, \hat{u}\right), v\right)_{\mathbb{R}^N} dt \quad \text{for all } v \in W^{1,p}, \end{aligned}$$

see (3.20) and (3.22). Therefore,

$$\begin{aligned} &a\left(\hat{u}'(t)\right)' + \frac{d}{dt}\nabla G(\hat{u}(t)) = y(t) + f(t, \hat{u}(t)) \quad \text{for a.a. } t \in T, \\ &\hat{u}(0) = \hat{u}(b),\ \hat{u}'(0) = \hat{u}'(b). \end{aligned}$$

We will be done if we can show that $y(t) \in A(\hat{u}(t))$ for a.a. $t \in T$.

Let $\hat{J}_{\lambda_n}(\hat{u}_n)(\cdot) = J_{\lambda_n}(\hat{u}_n(\cdot))$ for all $n \in \mathbb{N}$. From Proposition 2.1 and the chain rule for Sobolev functions we have that $\hat{J}_{\lambda_n}(\hat{u}_n) \in W^{1,2}$ for all $n \in \mathbb{N}$ and

$$\left\{\hat{J}_{\lambda_n}\left(\hat{u}_n\right)\right\}_{n\in\mathbb{N}} \subseteq W^{1,2} \text{ is bounded.}$$

So, we may assume that $\hat{J}_{\lambda_n}(\hat{u}_n) \xrightarrow{w} w$ in $W^{1,2}$ and because of the compact embedding $W^{1,2} \hookrightarrow C(T, \mathbb{R}^N)$,

$$\hat{J}_{\lambda_n}\left(\hat{u}_n\right) \to w \quad \text{in } C(T, \mathbb{R}^N). \tag{3.23}$$

We know that

$$\hat{J}_{\lambda_n}\left(\hat{u}_n\right) + \hat{\lambda}_n \hat{A}_{\lambda_n}\left(\hat{u}_n\right) = \hat{u}_n \quad \text{for all } n \in \mathbb{N},$$

which implies $w = \hat{u}$, see (3.23) and (3.22). Also, from (3.23) we see that

$$\hat{J}_{\lambda_n}\left(\hat{u}_n\right) \to \hat{u} \quad \text{in } C(T, \mathbb{R}^N). \tag{3.24}$$

Moreover, we have

$$\hat{A}_{\lambda_n}(u_n) \in \mathfrak{a}\left(\hat{J}_{\lambda_n}(u_n)\right) \quad \text{for all } n \in \mathbb{N}, \tag{3.25}$$

see Proposition 2.1. From Proposition 3.7 we know that $\mathfrak{a}$ is maximal monotone. So, the graph of $\mathfrak{a}$ is sequentially closed in $L^2(T, \mathbb{R}^N) \times L^2(T, \mathbb{R}^N)_{\mathrm{w}}$. From (3.20), (3.24) and (3.25) we have $y \in \mathfrak{a}(\hat{u})$. This means that

$$y(t) \in A\left(\hat{u}(t)\right) \quad \text{for a.a. } t \in T.$$

Therefore, $\hat{u} \in C^1(T, \mathbb{R}^N)$ is a solution of problem (1.1). □

When $D(A) = \mathbb{R}^N$ we can avoid the approximation by problem (3.15) and can also relax a little hypothesis H(f)(i).

Now, the hypotheses on the map A are the following.

H(A)': $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $D(A) = \mathbb{R}^N$ and $0 \in A(0)$.

Remark 3.10 In this case we know that A has nonempty, compact and convex values and as a multifunction it is upper semicontinuous from $\mathbb{R}^N$ into $\mathbb{R}^N$, see Papageorgiou–Winkert [9, Proposition 6.1.13].

The more general conditions on the perturbation $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ read as follows.

H(f)': $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

(i) for every $\eta > 0$ there exists $a_\eta \in L^1(T)_+$ such that

$$|f(t,x)| \leq a_\eta(t) \quad \text{for a.a. } t \in T \text{ and for all } |x| \leq \eta;$$

(ii) same as hypothesis H(f)(ii).

The method of the proof remains the same. Only since we work directly on the inclusion problem (1.1) and do not pass first from its single-valued approximation (3.15), we do not use Theorem 2.3, but its multivalued counterpart due to Bader [1]. Then we can have the following existence theorem.

Theorem 3.11 *If hypotheses H(a), H(G), H(A)′ and H(f)′ hold, then problem* (1.1) *admits a solution* $\hat{u} \in C^1(T, \mathbb{R}^N)$.

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