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Parametric nonlinear nonhomogeneous Neumann equations involving a nonhomogeneous differential operator

Said El Manouni · Nikolaos S. Papageorgiou · Patrick Winkert

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Abstract This work is concerned with the existence of solutions to parametric elliptic equations driven by a nonhomogeneous differential operator with a nonhomogeneous Neumann boundary condition. The assumptions on the operator involve the p-Laplacian, the (p, q)-Laplacian, and the generalized p-mean curvature differential operator. Based on variational tools combined with truncation and comparison techniques we prove the existence of at least three nontrivial solutions provided the parameter is sufficiently large whereby the first solution is positive, the second one is negative and the third one has changing sign (nodal).

Keywords Constant sign and nodal solutions · Nonlinear regularity · Critical point of mountain pass type · Extremal solutions · Local minimizers · Maximum principle

Mathematics Subject Classification (2010) 35J20 · 35J60 · 35J92 · 58E05

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < q \leq p$. We study the following nonlinear nonhomogeneous Neumann problem

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S. El Manouni · P. Winkert (✉)
Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany
e-mail: winkert@math.tu-berlin.de

S. El Manouni
e-mail: manouni@math.tu-berlin.de

N. S. Papageorgiou
Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece
e-mail: npapg@math.ntua.gr

$$
\begin{aligned}
-\operatorname{div} a(\nabla u) &= -\chi |u|^{p-2}u - f(x,u) \qquad \text{in} \quad \Omega, \\
\frac{\partial u}{\partial n_a} &= \lambda |u|^{q-2}u - h(x,u) \qquad \text{on} \quad \partial\Omega,
\end{aligned} \tag{1.1}
$$

where $\partial u/\partial n_a$ denotes the conormal derivative with respect to the mapping $a : \mathbb{R}^N \to \mathbb{R}^N$ which is supposed to be continuous and strictly monotone with $(p-1)$-growth. The nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are assumed to be Carathéodory functions being $(p-1)$-superlinear near $\pm\infty$ and bounded on bounded sets while χ, λ are real parameters to be specified. The aim of this paper is to establish the existence of at least three nontrivial solutions provided $\lambda > 0$ is sufficiently large depending on the first two eigenvalues of the negative q-Laplacian $-\Delta_q$ with Steklov boundary condition. In addition we give complete sign information of the solutions obtained, that is, the first solution is positive, the second one is negative and finally, the third one has changing sign.

Such results are known for quasilinear elliptic equations involving the p-Laplacian and were obtained by a number of authors in the last years with different methods. Without guarantee of completeness we refer to the papers of Abreu-Marcos do Ó-Medeiros [1], Fernández Bonder-Rossi [6], Fernández Bonder [7,8], Li-Li [14], Liu-Zheng [16], Martínez-Rossi [18], Winkert [23,25], Zhao-Zhao [27], and the references therein. To the best of our knowledge, the results presented here are the first one concerning multiplicity of solutions for equations involving a nonhomogeneous operator with nonhomogeneous Neumann boundary condition.

Our paper extends the results of Winkert [23] in different ways. On the one hand we can replace the p-Laplacian used in [23] by a more general nonhomogeneous operator and on the other hand we can drop a hypothesis on the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ required in [23], namely:

(H) There exists a number $\delta_f > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s} \geq 0$ for all $0 < |s| \leq \delta_f$ and for a.a. $x \in \Omega$.

Assumption (H) implies that the function f must change sign near zero. Now, we do not need this condition on f. It is also worth pointing out that we do not need differentiability, polynomial growth or some integral conditions on the mappings f and h. Our approach is based on variational methods coupled with truncation and comparison techniques.

2 Preliminaries and hypotheses

In this section we recall some basic facts about critical point theory which will be needed in Sect. 3. For this purpose, let X be a Banach space with norm $\|\cdot\|_X$ and denote by X^* its dual space equipped with the dual norm $\|\cdot\|_{X^*}$, that is

$$\|\xi\|_{X^*} = \sup\left\{\langle \xi, v\rangle_{(X^*,X)} : v \in X, \|v\|_X \leq 1\right\},$$

where $\langle\cdot,\cdot\rangle_{(X^*,X)}$ stands for the duality paring of (X^*, X).

Definition 2.1 The functional $\varphi \in C^1(X)$ fulfills the Palais-Smale condition at the level $c \in \mathbb{R}$ (the PS$_c$-condition for short) if every sequence $(u_n)_{n\geq 1} \subseteq X$ satisfying $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in X^*, admits a strongly convergent subsequence. We say that φ satisfies the Palais-Smale condition (the PS-condition for short) if it satisfies the PS$_c$-condition for every $c \in \mathbb{R}$.

This compactness-type condition on φ leads to a deformation theorem which is the main ingredient in the minimax theory of the critical values of φ. A basic result in that theory is the so-called mountain pass theorem.

Theorem 2.2 *If $\varphi \in C^1(X)$, $u_1, u_2 \in X$, $\|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: m_\rho$$

and φ satisfies the PS$_c$-condition where

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$$

with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$, then $c \geq m_\rho$ and c is a critical value of φ.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\begin{aligned} &\varphi^c = \{u \in X : \varphi(u) \leq c\} && \text{(the sublevel set of } \varphi \text{ at } c\text{)},\\ &K_\varphi = \{u \in X : \varphi'(u) = 0\} && \text{(the critical set of } \varphi\text{)},\\ &K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\} && \text{(the critical set of } \varphi \text{ at the level } c\text{)}. \end{aligned}$$

The following result is the so-called second deformation theorem (see, for example, Gasiński and Papageorgiou [11, p. 628]).

Theorem 2.3 *If $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \leq +\infty$, φ satisfies the PS$_c$-condition for every $c \in [a, b)$, φ has no critical values in (a, b) and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ, then there exists a continuous map $\hat{h} : [0,1] \times \left(\varphi^b \setminus K_\varphi^b\right) \to \varphi^b$ such that*

(a) $\hat{h}(0, u) = u$ for all $u \in \varphi^b \setminus K_\varphi^b$;
(b) $\hat{h}(1, \varphi^b \setminus K_\varphi^b) \subseteq \varphi^a$;
(c) $\hat{h}(t, u) = u$ for all $(t, u) \in [0,1] \times \varphi^a$;
(d) $\varphi(\hat{h}(t, u)) \leq \varphi(\hat{h}(s, u))$ for all $t, s \in [0,1]$, $s \leq t$, and all $u \in \varphi^b \setminus K_\varphi^b$.

By $L^p(\Omega)$ $\left(\text{or } L^p\left(\Omega; \mathbb{R}^N\right)\right)$, $L^p(\partial\Omega)$, and $W^{1,p}(\Omega)$ we denote the usual Lebesgue and Sobolev spaces with their norms $\|\cdot\|_{p,\Omega}$, $\|\cdot\|_{p,\partial\Omega}$, respectively, $\|\cdot\|_{1,p}$, which is given by

$$\|u\|_{1,p} = \left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p\right)^{\frac{1}{p}} \quad \text{for all} \quad u \in W^{1,p}(\Omega).$$

The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. In addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the ordered Banach space $C^1(\overline{\Omega})$ with norm $\|\cdot\|_{C^1(\overline{\Omega})}$ and its positive cone

$$C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\},$$

which has a nonempty interior given by

$$\operatorname{int}\left(C^1(\overline{\Omega})_+\right) = \left\{u \in C^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

Now let $\omega \in C^1(0,+\infty)$ be a function satisfying

$$0 < c_0 \leq \frac{t\omega'(t)}{\omega(t)} \leq c_1 \quad \text{and} \quad c_2 t^{p-1} \leq \omega(t) \leq c_3(1+t^{p-1})$$

for all $t > 0$ and with some constants $c_0, c_1, c_2, c_3 > 0$. The hypotheses on $a : \mathbb{R}^N \to \mathbb{R}^N$ read as follows.

H(a): $a(\xi) = a_0\left(\|\xi\|_{\mathbb{R}^N}\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \mapsto ta_0(t)$ is strictly increasing, $\lim_{t\to 0^+} ta_0(t) = 0$, and $\displaystyle\lim_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} > -1$;

(ii) $\|\nabla a(\xi)\|_{\mathbb{R}^N} \leq c_4 \dfrac{\omega\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}}$ for all $\xi \in \mathbb{R}^N\setminus\{0\}$ and some $c_4 > 0$;

(iii) $(\nabla a(\xi)y, y)_{\mathbb{R}^N} \geq \dfrac{\omega\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}} \|y\|^2_{\mathbb{R}^N}$ for all $\xi \in \mathbb{R}^N\setminus\{0\}$ and all $y \in \mathbb{R}^N$.

It is easy to see that condition H(a)(i) implies that $a \in C^1\left(\mathbb{R}^N\setminus\{0\}, \mathbb{R}^N\right) \cap C\left(\mathbb{R}^N, \mathbb{R}^N\right)$ and so, the assumptions in hypotheses H(a)(ii), (iii) are reasonable.

Let $G_0(t) = \int_0^t sa_0(s)ds$ and let $G(\xi) = G_0\left(\|\xi\|_{\mathbb{R}^N}\right)$ for all $\xi \in \mathbb{R}^N$. Then

$$\nabla G(\xi) = G_0'\left(\|\xi\|_{\mathbb{R}^N}\right)\frac{\xi}{\|\xi\|_{\mathbb{R}^N}} = a_0\left(\|\xi\|_{\mathbb{R}^N}\right)\xi = a(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^N\setminus\{0\},$$

which means that $G(\cdot)$ is the primitive of $a(\cdot)$. Obviously, $G(\cdot)$ is convex and since $G(0) = 0$ we have the estimate

$$G(\xi) \leq (a(\xi), \xi)_{\mathbb{R}^N} \quad \text{for all} \quad \xi \in \mathbb{R}^N. \tag{2.1}$$

The following lemma gives some basic properties of the mapping $a : \mathbb{R}^N \to \mathbb{R}^N$.

Lemma 2.4 *Under the hypotheses H(a) there holds*

(i) $\xi \to a(\xi)$ *is maximal monotone and strictly monotone;*

(ii) $\|a(\xi)\|_{\mathbb{R}^N} \leq c_5\left(1 + \|\xi\|^{p-1}_{\mathbb{R}^N}\right)$ *for all* $\xi \in \mathbb{R}^N$ *and some* $c_5 > 0$;

(iii) $(a(\xi), \xi)_{\mathbb{R}^N} \geq \frac{c_2}{p-1} \|\xi\|^p_{\mathbb{R}^N}$ *for all* $\xi \in \mathbb{R}^N$.

Taking into account Lemma 2.4 combined with (2.1) we infer the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.5 *If hypotheses H(a) hold, then*

$$\frac{c_2}{p(p-1)} \|\xi\|^p_{\mathbb{R}^N} \leq G(\xi) \leq c_6 \left(1 + \|\xi\|^p_{\mathbb{R}^N}\right) \quad \text{for all} \quad \xi \in \mathbb{R}^N \text{ and some } \quad c_6 > 0.$$

It should be mentioned that the operator $a : \mathbb{R}^N \to \mathbb{R}^N$ defined through hypotheses H(a) contains several interesting differential operators as special cases.

(i) Let $1 < p < \infty$ and let $a(\xi) = \|\xi\|^{p-2}_{\mathbb{R}^N} \xi$ with $1 < p < \infty$. Then $a(\cdot)$ represents the well-known p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div} \left(\|\nabla u\|^{p-2}_{\mathbb{R}^N} \nabla u\right) \quad \text{for all} \quad u \in W^{1,p}(\Omega).$$

The corresponding potential is given by $G(\xi) = \frac{1}{p} \|\xi\|^p_{\mathbb{R}^N}$ for all $\xi \in \mathbb{R}^N$.

(ii) Let $1 < q < p$ and let $a(\xi) = \|\xi\|^{p-2}_{\mathbb{R}^N} \xi + \|\xi\|^{q-2}_{\mathbb{R}^N} \xi$. Then $a(\cdot)$ becomes the (p, q)-differential operator defined by

$$\Delta_p u + \Delta_q u = \operatorname{div} \left(\|\nabla u\|^{p-2}_{\mathbb{R}^N} \nabla u\right) + \operatorname{div} \left(\|\nabla u\|^{q-2}_{\mathbb{R}^N} \nabla u\right)$$

for all $u \in W^{1,p}(\Omega)$. The associated potential is $G(\xi) = \frac{1}{p} \|\xi\|^p_{\mathbb{R}^N} + \frac{1}{q} \|\xi\|^q_{\mathbb{R}^N}$ for all $\xi \in \mathbb{R}^N$.

(iii) Let $1 < p < \infty$ and let $a(\xi) = \left(1 + \|\xi\|^2_{\mathbb{R}^N}\right)^{\frac{p-2}{2}} \xi$. In this case $a(\cdot)$ represents the generalized p-mean curvature differential operator which is defined by

$$\operatorname{div} \left[\left(1 + \|\nabla u\|^2_{\mathbb{R}^N}\right)^{\frac{p-2}{2}} \nabla u\right] \quad \text{for all} \quad u \in W^{1,p}(\Omega).$$

The potential is $G(\xi) = \frac{1}{p} \left[\left(1 + \|\xi\|^2_{\mathbb{R}^N}\right)^{\frac{p}{2}} - 1\right]$ for all $\xi \in \mathbb{R}^N$.

Now, let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$, $h_0 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying the subsequent growth conditions

$$|f_0(x, s)| \leq c_{f_0} \left(1 + |s|^{r_1 - 1}\right) \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$
$$|h_0(x, s)| \leq c_{h_0} \left(1 + |s|^{r_2 - 1}\right) \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R},$$

with $c_{f_0}, c_{h_0} > 0$ and $1 < r_1 < p^*$, $1 < r_2 < p_*$, where p^*, p_* denote the critical exponents of p given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases} \quad \text{and} \quad p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}.$$

Setting $F_0(x, s) = \int_0^s f_0(x, t)dt$, $H_0(x, s) = \int_0^s h_0(x, t)dt$ we define the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ through

$$\varphi_0(u) = \int_\Omega G(\nabla u)dx - \int_\Omega F_0(x, u)dx - \int_{\partial\Omega} H_0(x, u)d\sigma.$$

The following result concerning local minimizers is originally due to Brezis-Nirenberg [2] and was extended by García Azorero-Peral Alonso-Manfredi [9], Motreanu-Papageorgiou [20], Winkert [24], and Khan-Motreanu [12].

Proposition 2.6 *Let the assumptions in H(a) be satisfied. If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all} \quad h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\|_{1,p} \leq \rho_1.$$

Proof The theorem follows directly from the abstract result obtained by Khan-Motreanu [12]. Indeed, let $X = C^1(\overline{\Omega})$, $Y = W^{1,p}(\Omega)$, and let

$$J(u) = \int_\Omega G(\nabla u)dx \quad \text{and} \quad E(u) = \int_\Omega F_0(x, u)dx + \int_{\partial\Omega} H_0(x, u)d\sigma.$$

Setting

$$\Phi(u) = \left(\|u\|_{r_1,\Omega}^{r_1} + \|u\|_{r_2,\partial\Omega}^{r_2}\right)^{\max(r_1,r_2)},$$

it is straightforward to verify that the assumptions in [12, Theorem 2.1] are satisfied. This completes the proof. □

Now, let $A : W^{1,p}(\Omega) \to \left(W^{1,p}(\Omega)\right)^*$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega (a(\nabla u), \nabla v)_{\mathbb{R}^N}\, dx \quad \text{for all } u, v \in W^{1,p}(\Omega). \tag{2.2}$$

The next proposition gives the main properties of A (see, for example, Gasiński-Papageorgiou [10, p. 562]).

Proposition 2.7 *Let hypotheses H(a) be satisfied. Then* $A : W^{1,p}(\Omega) \to \left(W^{1,p}(\Omega)\right)^*$ *defined by* (2.2) *is continuous, monotone (hence maximal monotone) and of type* $(S)_+$*, i.e., if* $u_n \rightharpoonup u$ *in* $W^{1,p}(\Omega)$ *and* $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0$*, then* $u_n \to u$ *in* $W^{1,p}(\Omega)$.

Given $1 < r < \infty$, we denote by $\Delta_r : W^{1,r}(\Omega) \to \left(W^{1,r}(\Omega)\right)^*$ the r-Laplacian which is defined by

$$\langle \Delta_r u, v\rangle = \int_\Omega \|\nabla u\|_{\mathbb{R}^N}^{r-2} (\nabla u, \nabla v)_{\mathbb{R}^N}\, dx \quad \text{for all} \quad u, v \in W^{1,r}(\Omega).$$

If $r = 2$, then $\Delta_r = \Delta$ becomes the well-known Laplace operator and we have $\Delta \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$, where $\mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ denotes the vector space of all bounded linear operators from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

A main role in our treatment plays the spectrum of the r-Laplacian with a Steklov boundary condition. To this end, we consider the following eigenvalue problem

$$\begin{aligned} -\Delta_r u &= -|u|^{r-2}u \qquad \text{in} \quad \Omega, \\ \|\nabla u\|_{\mathbb{R}^N}^{r-2} \frac{\partial u}{\partial n} &= \hat{\lambda}|u|^{r-2}u \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.3}$$

where $\partial u/\partial n$ is the outer normal derivative of u at $\partial\Omega$. A number $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $\left(-\Delta_r, W^{1,r}(\Omega)\right)$ if problem (2.3) admits a nontrivial weak solution $\hat{u} \in W^{1,p}(\Omega)$ which is called an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. The set of eigenvalues is denoted by $\hat{\sigma}(r)$ which has a smallest element $\hat{\lambda}_1(r)$. The spectrum of (2.3) were intensively studied by Lê [13] and Martínez-Rossi [17] whereby the main facts read as follows:

- $\hat{\lambda}_1(r)$ is positive, isolated, and simple;
- $$\hat{\lambda}_1(r) = \inf_{u\in W^{1,p}(\Omega)} \left\{ \int_\Omega \|\nabla u\|_{\mathbb{R}^N}^r dx + \int_\Omega |u|^r dx : \|u\|_{r,\partial\Omega}^r = 1 \right\};$$
- $\hat{\sigma}(r)$ is closed.

We further point out that every eigenfunction corresponding to the first eigenvalue $\hat{\lambda}_1(r)$ does not change sign in $\overline{\Omega}$. In fact it turns out that every eigenfunction associated to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(r)$ changes sign on $\partial\Omega$.

In what follows we denote by $\hat{u}_1(r)$ the normalized (i.e., $\|\hat{u}_1(r)\|_{r,\partial\Omega} = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1(r)$. As shown in Lê [13], thanks to the nonlinear regularity theory and the nonlinear maximum principle, we can suppose that $\hat{u}_1(r) \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Additionally, due to the fact that $\hat{\lambda}_1(r)$ is isolated, the second eigenvalue $\hat{\lambda}_2(r)$ is well-defined by

$$\hat{\lambda}_2(r) = \inf\left[\hat{\lambda} \in \hat{\sigma}(r) : \hat{\lambda} > \hat{\lambda}_1(r)\right].$$

Now, let $\partial B_1^{r,\partial\Omega} = \{u \in L^r(\partial\Omega) : \|u\|_{r,\partial\Omega} = 1\}$ and $S_r = W^{1,r}(\Omega) \cap \partial B_1^{r,\partial\Omega}$. Then, due to Martínez-Rossi [19], we have the following variational characterization of $\hat\lambda_2(r)$.

Proposition 2.8 *There holds*

$$\hat\lambda_2(r) = \inf_{\hat\gamma\in\hat\Gamma(r)} \max_{-1\le t\le 1} \left[\int_\Omega \|\nabla\hat\gamma(t)\|_{\mathbb{R}^N}^r dx + \int_\Omega |\hat\gamma(t)|^r dx \right],$$

where $\hat\Gamma(r) = \left\{\hat\gamma \in C([-1,1], S_r) : \hat\gamma(-1) = -\hat u_1(r), \hat\gamma(1) = \hat u_1(r)\right\}$.

Recall that if a functional satisfies the PS-condition (or the C-condition) and it is bounded below, then it is coercive (see Čaklović-Li-Willem [3] and Gasiński-Papageorgiou [11, p. 614]). The converse assertion is in general not true, but in our setting we can give a positive answer.

Indeed, let $\hat f : \Omega\times\mathbb{R} \to \mathbb{R}$, $\hat h : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying

$$|\hat f(x,s)| \le c_{\hat f}\left(1 + |s|^{r_1-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$
$$|\hat h(x,s)| \le c_{\hat h}\left(1 + |s|^{r_2-1}\right) \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R},$$

with $c_{\hat f}, c_{\hat h} > 0$, $1 < r_1 < p^*$, and $1 < r_2 < p_*$. We set $\hat F(x,s) = \int_0^s \hat f(x,t)dt$, $\hat H(x,s) = \int_0^s \hat h(x,t)dt$ and consider the C^1-functional $\hat\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat\varphi(u) = \int_\Omega G(\nabla u)dx - \int_\Omega \hat F(x,u)dx - \int_{\partial\Omega} \hat H(x,u)d\sigma.$$

Proposition 2.9 *If $\hat\varphi$ is coercive, then it satisfies the PS-condition.*

Proof Let $(u_n)_{n\ge1} \subseteq W^{1,p}(\Omega)$ be a PS-sequence, that is

$$|\hat\varphi(u_n)| \le \hat M \quad \text{for some } \hat M > 0, \text{ for all } n \ge 1, \tag{2.4}$$
$$\left(\hat\varphi\right)'(u_n) \to 0 \quad \text{in } \left(W^{1,p}(\Omega)\right)^*. \tag{2.5}$$

Since $\hat\varphi$ is coercive and due to (2.4) we easily verify that $(u_n)_{n\ge1}$ is bounded in $W^{1,p}(\Omega)$. Because of that we may assume

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega) \quad u_n \to u \quad \text{in } L^p(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\partial\Omega). \tag{2.6}$$

The assertion in (2.5) implies that

$$\left| \int_\Omega (a(\nabla u_n), \nabla v)_{\mathbb{R}^N}\, dx - \int_\Omega \hat f(x,u_n)v dx - \int_{\partial\Omega} \hat h(x,u_n)v d\sigma \right| \le \varepsilon_n \|v\|_{1,p},$$

for all $v \in W^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. Now, choosing $v = u_n - u$, passing to the limit as $n \to \infty$, and using the convergence properties in (2.6), we have

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = \lim_{n\to\infty} \int_\Omega (a(\nabla u_n), \nabla (u_n - u))_{\mathbb{R}^N}\, dx = 0,$$

which by the $(S)_+$-property of A (see Proposition 2.7) gives $u_n \to u$ in $W^{1,p}(\Omega)$ proving that $\hat{\varphi}$ satisfies the PS-condition. □

Finally, for $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. Recall that

$$u^\pm \in W^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

The Lebesgue measure on $\mathbb{R}^N$ is given by $|\cdot|_N$.

3 Three solutions depending on Steklov eigenvalues

We are now interested in the existence of weak solutions to Eq. (1.1) depending on Steklov eigenvalues of the q-Laplacian with $1 < q \le p < \infty$. In order to prove this we need some additional assumptions on the map $a : \mathbb{R}^N \to \mathbb{R}^N$.

H(a)$_1$: $a(\xi) = a_0\left(\|\xi\|_{\mathbb{R}^N}\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, hypotheses H(a)$_1$(i)–(iii) are the same as the corresponding hypotheses H(a)(i)–(iii) and

(iv) if $G_0(t) = \int_0^t s a_0(s) ds$ for all $t > 0$, then $t \mapsto G_0\left(t^{\frac{1}{q}}\right)$ is convex in $(0, +\infty)$ and

$$\limsup_{t\to 0^+} \frac{G_0(t)}{t^q} < +\infty.$$

Remark 3.1 The examples presented in Sect. 2 still satisfy hypotheses H$(a)_1$. Note that by hypothesis H(a)$_1$(iv) we find $c_7 > 0$ such that

$$G(\xi) \le c_7 \left(\|\xi\|^q_{\mathbb{R}^N} + \|\xi\|^p_{\mathbb{R}^N}\right) \quad \text{for all} \quad \xi \in \mathbb{R}^N. \tag{3.1}$$

The hypotheses on the Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ and the number χ read as follows.

(H1) f is bounded on bounded sets;
(H2) $\lim_{s\to\pm\infty} \dfrac{f(x,s)}{|s|^{p-2}s} = +\infty$ uniformly for a.a. $x \in \Omega$;
(H3) $\lim_{s\to 0} \dfrac{f(x,s)}{|s|^{p-2}s} = 0$ uniformly for a.a. $x \in \Omega$;
(H4) h is bounded on bounded sets;
(H5) $\lim_{s\to\pm\infty} \dfrac{h(x,s)}{|s|^{p-2}s} = +\infty$ uniformly for a.a. $x \in \partial\Omega$;

(H6) $\lim_{s\to 0} \dfrac{h(x,s)}{|s|^{q-2}s} = 0$ uniformly for a.a. $x \in \partial\Omega$;

(H7) h satisfies the condition

$$|h(x_1, s_1) - h(x_2, s_2)| \le L\Big[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\Big],$$

for all pairs (x_1, s_1), (x_2, s_2) in $\partial\Omega \times [-K, K]$, where K is a positive constant and $\alpha \in (0, 1]$;

(H8) χ is a real fixed number such that

$$0 < \chi \begin{cases} < +\infty & \text{if } q < p \\ \le 2pc_7 & \text{if } q = p \end{cases},$$

where c_7 is the positive constant given in Remark 3.1.

Remark 3.2 Note that hypothesis (H7) is needed for the usage of the C^1-regularity results of Lieberman [15]. It is obvious that $s \mapsto \lambda|s|^{q-2}s$ fulfills condition (H7) for $\lambda > 0$ and $1 < q \le p$. We also point out that no growth condition is imposed on f, h and thanks to (H3), (H6) we easily verify that $f(x, 0) = h(x, 0) = 0$ for a.a. $x \in \Omega$, resp., $x \in \partial\Omega$. Hence, $u = 0$ is a solution of problem (1.1).

A function $u \in W^{1,p}(\Omega)$ is said to be a (weak) solution of (1.1) if it satisfies the equation

$$\int_\Omega (a(\nabla u), \nabla\varphi)_{\mathbb{R}^N}\, dx = \int_\Omega \Big(-\chi|u|^{q-2}u - f(x,u)\Big)\varphi dx + \int_{\partial\Omega} \Big(\lambda|u|^{q-2}u - h(x,u)\Big)\varphi d\sigma$$

for all test functions $\varphi \in W^{1,p}(\Omega)$ while $d\sigma$ denotes the usual $(N-1)$-surface measure.

The conditions in (H2), (H5) imply the existence of constants M_1, $M_2 = M_2(\lambda) > 1$ such that

$$\begin{aligned} f(x,s)s &\ge |s|^p \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \ge M_1, \\ h(x,s)s &\ge \lambda|s|^p \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \ge M_2. \end{aligned} \tag{3.2}$$

Let $M_3 = \max(M_1, M_2)$. Taking $\overline{u} \equiv \varsigma \in [M_3, +\infty)$ and applying (3.2), $q \le p$, and $M_3 > 1$, we conclude

$$0 \ge -f(x, \overline{u}) \quad \text{a.e. in } \Omega \quad \text{and} \quad 0 \ge \lambda\overline{u}^{q-1} - h(x, \overline{u}) \quad \text{a.e. in} \quad \partial\Omega. \tag{3.3}$$

In the same way, choosing $\underline{v} \equiv -\varsigma$, we obtain

$$0 \le -f(x, \underline{v}) \quad \text{a.e. in } \Omega \quad \text{and} \quad 0 \le \lambda|\underline{v}|^{q-2}\underline{v} - h(x, \underline{v}) \quad \text{a.e. in} \quad \partial\Omega.$$

Now, we introduce the truncation functions

$$
\begin{aligned}
b^+(x,s) &= \begin{cases} 0 & \text{if } s < 0 \\ -f(x,s) & \text{if } 0 \le s \le \overline{u} \\ -f(x,\overline{u}) & \text{if } \overline{u} < s \end{cases}, \\
b^+_\lambda(x,s) &= \begin{cases} 0 & \text{if } s < 0 \\ \lambda s^{q-1} - h(x,s) & \text{if } 0 \le s \le \overline{u} \\ \lambda \overline{u}^{q-1} - h(x,\overline{u}) & \text{if } \overline{u} < s \end{cases},
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
b^-(x,s) &= \begin{cases} -f(x,\underline{v}) & \text{if } s < \underline{v} \\ -f(x,s) & \text{if } \underline{v} \le s \le 0 \\ 0 & \text{if } 0 < s \end{cases}, \\
b^-_\lambda(x,s) &= \begin{cases} \lambda|\underline{v}|^{q-2}\underline{v} - h(x,\underline{v}) & \text{if } s < \underline{v} \\ \lambda|s|^{q-2}s - h(x,s) & \text{if } \underline{v} \le s \le 0 \\ 0 & \text{if } 0 < s \end{cases},
\end{aligned}
$$

which are well-known to be Carathéodory functions. Setting $B^\pm(x,s) = \int_0^s b^\pm(x,t)dt$, $B^\pm_\lambda(x,s) = \int_0^s b^\pm_\lambda(x,t)dt$, we consider the C^1-functionals $\varphi^\pm_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$
\varphi^\pm_\lambda(u) = \int_\Omega G(\nabla u)dx + \frac{\chi}{p}\int_\Omega |u|^p dx - \int_\Omega B^\pm(x,u)dx - \int_{\partial\Omega} B^\pm_\lambda(x,u)d\sigma.
$$

Furthermore, we write $F(x,s) = \int_0^s f(x,t)dt$ and $H(x,s) = \int_0^s h(x,t)dt$. Recall that $\hat{u}_1(q) \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ denotes the normalized eigenfunction (i.e. $\|\hat{u}_1(q)\|_{q,\partial\Omega} = 1$) corresponding to the first eigenvalue $\hat{\lambda}_1(q)$ of the Steklov eigenvalue problem of the q-Laplacian given in (2.3).

We start with the existence of constant sign solutions to problem (1.1) provided $\lambda > 0$ is sufficiently large.

Proposition 3.3 *Let the assumptions in H(a)$_1$ and (H1)–(H8) be satisfied and suppose that*

$$
\lambda > \begin{cases} qc_7\hat{\lambda}_1(q) & \textit{if } q < p \\ 2pc_7\hat{\lambda}_1(p) & \textit{if } q = p \end{cases}
\tag{3.5}
$$

with the positive constant c_7 given in Remark 3.1. Then problem (1.1) has at least two nontrivial constant sign solutions

$$
u_0 \in [0,\overline{u}] \cap \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \textit{and} \quad v_0 \in [\underline{v},0] \cap \left(-\operatorname{int}\left(C^1(\overline{\Omega})_+\right)\right).
$$

Proof Let us begin the proof with the existence of the positive solution. By means of the truncation in (3.4) standard arguments ensure that φ_λ^+ is coercive and sequentially weakly lower semicontinuous. Therefore, the Weierstrass theorem yields the existence of $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda^+(u_0) = \inf\left[\varphi_\lambda^+(u) : u \in W^{1,p}(\Omega)\right] = m_\lambda^+. \tag{3.6}$$

Given $\varepsilon_1, \varepsilon_2 > 0$, from Hypotheses (H3), (H6) we find $\delta_1 = \delta_1(\varepsilon_1), \delta_2 = \delta_2(\varepsilon_2) \in (0, \overline{u})$ such that

$$\begin{aligned} F(x,s) &\le \frac{\varepsilon_1}{p}|s|^p \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \le \delta_1,\\ H(x,s) &\le \frac{\varepsilon_2}{q}|s|^q \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } |s| \le \delta_2. \end{aligned} \tag{3.7}$$

Let $\delta := \min(\delta_1, \delta_2)$. Since $\hat{u}_1(q) \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, we may choose $t \in (0,1)$ small enough such that $t\hat{u}_1(q)(x) \in [0,\delta]$ for all $x \in \overline{\Omega}$. Because of (3.1), (3.4), (3.7) along with $\|\hat{u}_1(q)\|_{q,\partial\Omega} = 1$ and $\delta < \overline{u}$ it follows

$$\begin{aligned} \varphi_\lambda^+(t\hat{u}_1(q)) &= \int_\Omega G(\nabla(t\hat{u}_1(q)))dx + \frac{\chi}{p}\int_\Omega |t\hat{u}_1(q)|^p dx - \int_\Omega B^+(x, t\hat{u}_1(q))dx\\ &\quad - \int_{\partial\Omega} B_\lambda^+(x, t\hat{u}_1(q))d\sigma\\ &= \int_\Omega G(\nabla(t\hat{u}_1(q)))dx + \frac{t^p\chi}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p + \int_\Omega F\left(x, t\hat{u}_1(q)\right)dx\\ &\quad - \frac{\lambda t^q}{q} + \int_{\partial\Omega} H\left(x, t\hat{u}_1(q)\right)d\sigma\\ &\le c_7\left(t^q\|\nabla(\hat{u}_1(q))\|_{q,\Omega}^q + t^p\|\nabla(\hat{u}_1(q))\|_{p,\Omega}^p\right) + \frac{t^p\chi}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p \qquad (3.8)\\ &\quad + \frac{\varepsilon_1 t^p}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p - \frac{\lambda t^q}{q} + \frac{\varepsilon_2 t^q}{q}\\ &\le c_7\left(-t^q\|\hat{u}_1(q)\|_{q,\Omega}^q + t^q\hat{\lambda}_1(q) + t^p\|\nabla(\hat{u}_1(q))\|_{p,\Omega}^p\right)\\ &\quad + \frac{t^p\chi}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p + \frac{\varepsilon_1 t^p}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p - \frac{\lambda t^q}{q} + \frac{\varepsilon_2 t^q}{q}\\ &= t^q\left(\frac{-c_7 q}{q}\right)\|\hat{u}_1(q)\|_{q,\Omega}^q + t^q\left(\frac{qc_7\hat{\lambda}_1(q) - \lambda + \varepsilon_2}{q}\right)\\ &\quad + t^p\left(c_7\|\nabla(\hat{u}_1(q))\|_{p,\Omega}^p + \frac{\chi + \varepsilon_1}{p}\|\hat{u}_1(q)\|_{p,\Omega}^p\right). \end{aligned}$$

If $q < p$ we may choose $\varepsilon_1, \varepsilon_2$ such that

$$0 < \varepsilon_1 < \infty \quad \text{and} \quad 0 < \varepsilon_2 < \lambda - qc_7\hat{\lambda}_1(q)$$

(see (3.5)), then (3.8) becomes

$$\varphi_\lambda^+(t\hat{u}_1(q)) \leq -t^q M_4 + t^p M_5 \tag{3.9}$$

with some $M_4, M_5 > 0$. Since $q < p$, (3.9) implies

$$\varphi_\lambda^+(t\hat{u}_1(q)) < 0 \quad \text{for all sufficiently small } t > 0. \tag{3.10}$$

If $q = p$, (3.8) reduces to

$$\begin{aligned} \varphi_\lambda^+(t\hat{u}_1(p)) \leq t^p \left(\frac{-2pc_7 + \chi + \varepsilon_1}{p} \right) \|\hat{u}_1(p)\|_{p,\Omega}^p \\ + t^p \left(\frac{2pc_7\hat{\lambda}_1(p) - \lambda + \varepsilon_2}{p} \right). \end{aligned} \tag{3.11}$$

If $2pc_7 > \chi$ we may choose

$$0 < \varepsilon_1 < 2pc_7 - \chi \quad \text{and} \quad 0 < \varepsilon_2 < \lambda - 2pc_7\hat{\lambda}_1(p)$$

(see (H8) and (3.5)) to obtain again (3.10). Finally, if $2pc_7 = \chi$, (3.11) becomes

$$\varphi_\lambda^+(t\hat{u}_1(p)) \leq t^p \left(\frac{\varepsilon_1}{p} \|\hat{u}_1(p)\|_{p,\Omega}^p + \frac{2c_7 p\hat{\lambda}_1(p) - \lambda + \varepsilon_2}{p} \right).$$

Choosing $0 < \varepsilon_2 < \lambda - 2pc_7\hat{\lambda}_1(p)$ we find a constant $M_6 = M_6(\lambda) > 0$ such that

$$\varphi_\lambda^+(t\hat{u}_1(p)) \leq t^p \left(\frac{\varepsilon_1}{p} \|\hat{u}_1(p)\|_{p,\Omega}^p - M_6 \right).$$

Taking $0 < \varepsilon_1 < \frac{M_6 p}{\|\hat{u}_1(p)\|_{p,\Omega}^p}$ provides inequality (3.10) again in this case. In summary the choices of $\varepsilon_1, \varepsilon_2$ above lead to (see also (3.6))

$$\varphi_\lambda^+(u_0) < 0 = \varphi_\lambda^+(0),$$

implying $u_0 \not\equiv 0$. Moreover, as u_0 is the global minimizer of φ_λ^+, there holds $\left(\varphi_\lambda^+\right)'(u_0) = 0$ meaning that

$$\begin{aligned} &\int_\Omega (a(\nabla u_0), \nabla \varphi)_{\mathbb{R}^N}\, dx + \chi \int_\Omega |u_0|^{p-2} u_0 \varphi dx \\ &\quad = \int_\Omega b^+(x, u_0)\varphi dx + \int_{\partial\Omega} b_\lambda^+(x, u_0)\varphi d\sigma, \end{aligned} \tag{3.12}$$

for all $\varphi \in W^{1,p}(\Omega)$. We take $\varphi = -u_0^- \in W^{1,p}(\Omega)$ as test function in (3.12) and by virtue of Lemma 2.4(iii) in combination with the definition of the truncations (see (3.4)), we obtain

$$\min\left(\frac{c_2}{p-1}, \chi\right)\left(\|\nabla u_0^-\|_{p,\Omega}^p + \|u_0^-\|_{p,\Omega}^p\right) \le 0,$$

which means that $u_0 \ge 0$. Choosing $\varphi = (u_0 - \overline{u})^+ \in W^{1,p}(\Omega)$ in (3.12) and making use of (3.3) as well as (3.4), it follows

$$\begin{aligned} &\int_\Omega \left(a(\nabla u_0), \nabla (u_0 - \overline{u})^+\right)_{\mathbb{R}^N} dx + \chi \int_\Omega u_0^{p-1} (u_0 - \overline{u})^+\, dx \\ &= \int_\Omega b^+(x, u_0)(u_0 - \overline{u})^+ dx + \int_{\partial\Omega} b_\lambda^+(x, u_0)(u_0 - \overline{u})^+ d\sigma \\ &= \int_\Omega (-f(x, \overline{u}))\,(u_0 - \overline{u})^+ dx + \int_{\partial\Omega} \left(\lambda \overline{u}^{q-1} - h(x, \overline{u})\right)(u_0 - \overline{u})^+ d\sigma \\ &\le 0. \end{aligned}$$

This gives, due to Lemma 2.4(iii),

$$\begin{aligned} 0 &\ge \int_{\{u_0 > \overline{u}\}} (a(\nabla u_0), \nabla u_0)_{\mathbb{R}^N}\, dx + \chi \int_{\{u_0 > \overline{u}\}} u_0^{p-1} (u_0 - \overline{u})\, dx \\ &\ge \frac{c_2}{p-1} \int_{\{u_0 > \overline{u}\}} \|\nabla u_0\|_{\mathbb{R}^N}^p\, dx + \int_{\{u_0 > \overline{u}\}} (u_0 - \overline{u})^p\, dx \\ &\ge \int_\Omega \left((u_0 - \overline{u})^+\right)^p dx \\ &\ge 0. \end{aligned}$$

Hence $|\{u_0 > \overline{u}\}|_N = 0$, that is $u_0 \le \overline{u}$. We conclude that $u_0 \in [0, \overline{u}]$ with $u_0 \not\equiv 0$. Then, by means of the truncations in (3.4), Eq. (3.12) becomes

$$\int_\Omega (a(\nabla u_0), \nabla\varphi)_{\mathbb{R}^N}\, dx + \chi \int_\Omega u_0^{p-1}\varphi dx = \int_\Omega (-f(x, u_0))\, \varphi dx + \int_{\partial\Omega} \left(\lambda u_0^{q-1} - h(x, u_0)\right) \varphi d\sigma$$

which means that $u_0 \in W^{1,p}(\Omega)$ solves the problem

$$\begin{aligned} -\operatorname{div} a(\nabla u_0) &= -\chi u_0^{p-1} - f(x, u_0) \qquad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n_a} &= \lambda u_0^{q-1} - h(x, u_0) \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.13}$$

Since $u_0 \in [0, \overline{u}]$ we have $u_0 \in L^\infty(\Omega)$ (see also Winkert-Zacher [26, Corollary 1.2]) and from the regularity results of Lieberman [15] it follows that $u_0 \in C^1(\overline{\Omega})\backslash\{0\}$. Taking into account (H1), (H3) we find a constant $M_7 > 0$ such that

$$f(x, s) \le M_7 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \le s \le \overline{u}. \tag{3.14}$$

Combining (3.13) and (3.14) yields

$$\operatorname{div} a(\nabla u_0) \le (\chi + M_7)\, u_0^{p-1} \quad \text{a.e. in} \quad \Omega.$$

Now, we may apply the strong maximum principle (see Pucci-Serrin [21, Theorem 2.5.1]) to obtain $u_0(x) > 0$ for all $x \in \Omega$.

Let $x_0 \in \partial\Omega$ be such that $u_0(x_0) = 0$. Applying the boundary point lemma (see again Pucci-Serrin [21, Theorem 5.5.1]) gives

$$\frac{\partial u_0}{\partial n_a}(x_0) = a_0 \left(\|\nabla u_0\|_{\mathbb{R}^N}\right) \frac{\partial u_0}{\partial n}(x_0) < 0, \tag{3.15}$$

where $(\partial u_0/\partial n)(x_0)$ stands for the outer normal derivative of u_0 at $x_0 \in \partial\Omega$. Since $h(x_0, u_0(x_0)) = h(x_0, 0) = 0$ we get a contradiction from (3.13) and (3.15). Hence, $u_0(x) > 0$ for all $x \in \overline{\Omega}$ and consequently $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. That finishes the first part of the theorem.

The second assertion can be shown similarly using φ_λ^- instead of φ_λ^+ to obtain the existence of a nontrivial negative solution $v_0 \in [\underline{v}, 0] \cap \left(-\operatorname{int}\left(C^1(\overline{\Omega})_+\right)\right)$. □

Now we are going to prove the existence of extremal constant sign solutions of (1.1) provided $\lambda > 0$ is large enough as before.

To this end, let $\mathcal{S}_+(\lambda)(\mathcal{S}_-(\lambda))$ be the set of all nontrivial positive (negative) solutions of problem (1.1). Thanks to the monotonicity of a (see Lemma 2.4(i)) one can show that $\mathcal{S}_+(\lambda)$ $(\mathcal{S}_-(\lambda))$ is downward (upward) directed, that means if $u_1, u_2 \in \mathcal{S}_+(\lambda)$

$(\mathcal{S}_-(\lambda))$, then there is an element $\hat{v} \in \mathcal{S}_+(\lambda)$ $(\mathcal{S}_-(\lambda))$ such that $\hat{v} \leq (\geq) u_1$, $\hat{v} \leq (\geq) u_2$. Therefore, without loss of generality, we can focus on the sets

$$\hat{\mathcal{S}}_+(\lambda) = \mathcal{S}_+(\lambda) \cap [0, \overline{u}], \qquad \hat{\mathcal{S}}_-(\lambda) = \mathcal{S}_-(\lambda) \cap [\underline{v}, 0].$$

As a consequence of Proposition 3.3 we know that both sets are nonempty, i.e., $\hat{S}_+(\lambda) \neq \emptyset$ and $\hat{S}_-(\lambda) \neq \emptyset$. We can further suppose, without loss of generality, that

$$\begin{aligned} |f(x,s)| &\leq M_8 \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}, \\ |h(x,s)| &\leq M_9 \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R}, \end{aligned} \tag{3.16}$$

with positive constants M_8, M_9 which can be seen by truncation of $f(x,\cdot), h(x,\cdot)$ at $\underline{v}$ (from below) and $\overline{u}$ (from above) combined with (H1), (H4). Then, taking into account hypotheses (H3), (H6) along with (3.16) we find for given $\varepsilon_1, \varepsilon_2 > 0$ and $r_1 \in (p, p^*)$, $r_2 \in (p, p_*)$ numbers $M_{10} = M_{10}(\varepsilon_1, r_1), M_{11} = M_{11}(\varepsilon_2, r_2) > 0$ such that

$$\begin{aligned} f(x,s)s &\leq \varepsilon_1 |s|^p + M_{10}|s|^{r_1} \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}, \\ h(x,s)s &\leq \varepsilon_2 |s|^q + M_{11}|s|^{r_2} \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}. \end{aligned} \tag{3.17}$$

In order to prove the existence of a smallest positive and a greatest negative solution to (1.1) we will consider an auxiliary problem. To this end, let $\lambda > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 \in (0, \lambda)$ and consider the subsequent equation

$$\begin{aligned} -\operatorname{div}(a(\nabla u)) &= -(\chi + \varepsilon_1)|u|^{p-2}u - M_{10}|u|^{r_1-2}u \qquad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n_a} &= (\lambda - \varepsilon_2)|u|^{q-2}u - M_{11}|u|^{r_2-2}u \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.18}$$

We are going to prove the uniqueness of constant sign solutions of problem (3.18).

Proposition 3.4 *Let hypotheses H(a)$_1$ and (H8) be satisfied and suppose*

$$\lambda > \begin{cases} qc_7\hat{\lambda}_1(q) & \text{if } q < p, \\ 2pc_7\hat{\lambda}_1(p) & \text{if } q = p. \end{cases}$$

Then, problem (3.18) *has a unique positive solution* $u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ *and a unique negative solution* $v_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

Proof Due to the oddness of (3.18) it suffices to prove the existence of a unique positive solution $u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, the existence of a unique negative solution follows directly by setting $v_* = -u_* \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

Let $\Psi_\lambda^+ : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\Psi_\lambda^+(u) = \int_\Omega G(\nabla u)dx + \frac{\chi}{p}\|u\|_{p,\Omega}^p + \frac{\varepsilon_1}{p}\|u^+\|_{p,\Omega}^p + \frac{M_{10}}{r_1}\|u^+\|_{r_1,\Omega}^{r_1} - \frac{\lambda - \varepsilon_2}{q}\|u^+\|_{q,\partial\Omega}^q + \frac{M_{11}}{r_2}\|u^+\|_{r_2,\partial\Omega}^{r_2}.$$

Since $q \le p < r_1 < p^*, q \le p < r_2 < p_*$ we note that Ψ_λ^+ is coercive and sequentially weakly lower semicontinuous. Hence, its global minimizer $u_* \in W^{1,p}(\Omega)$ exists and as in Proposition 3.3, the choice of $\lambda > 0$ yields

$$\Psi_\lambda^+(u_*) < 0 = \Psi_\lambda^+(0)$$

guaranteeing $u_* \neq 0$. Since u_* is a global minimizer of Ψ_λ^+ we have $\left(\psi_\lambda^+\right)'(u_*) = 0$, that is

$$\begin{aligned} &\int_\Omega (a(\nabla u_*), \nabla\varphi)_{\mathbb{R}^N}\, dx + \chi\int_\Omega |u_*|^{p-2}u_*\varphi dx \\ &\quad = -\varepsilon_1\int_\Omega \left(u_*^+\right)^{p-1}\varphi dx - M_{10}\int_\Omega \left(u_*^+\right)^{r_1-1}\varphi dx \\ &\qquad + (\lambda - \varepsilon_2)\int_{\partial\Omega} \left(u_*^+\right)^{q-1}\varphi d\sigma - M_{11}\int_{\partial\Omega} \left(u_*^+\right)^{r_2-1}\varphi d\sigma, \end{aligned} \tag{3.19}$$

for all $\varphi \in W^{1,p}(\Omega)$. Taking $\varphi = -u_*^- \in W^{1,p}(\Omega)$ and applying Lemma 2.4(iii), we get $u_* \ge 0$ (cf. the proof of Proposition 3.3). Then, (3.19) becomes

$$\begin{aligned} &\int_\Omega (a(\nabla u_*), \nabla\varphi)_{\mathbb{R}^N}\, dx + \chi\int_\Omega u_*^{p-1}\varphi dx \\ &\quad = -\varepsilon_1\int_\Omega u_*^{p-1}\varphi dx - M_{10}\int_\Omega u_*^{r_1-1}\varphi dx - (\lambda-\varepsilon_2)\int_{\partial\Omega} u_*^{q-1}\varphi d\sigma - M_{11}\int_{\partial\Omega} u_*^{r_2-1}\varphi d\sigma, \end{aligned}$$

meaning that u_* is a nontrivial positive solution of (3.18). Moreover, the nonlinear regularity theory (see Winkert-Zacher [26] and Lieberman [15]) combined with the nonlinear maximum principle (see Pucci-Serrin [21]) yields $u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ (similar to the proof of Proposition 3.3).

We are done with the proof provided u_* is shown to be the unique positive solution of (3.18). Let $\Upsilon_+ : L^1(\Omega) \to \mathbb{R} \cup \{\infty\}$ be the integral functional defined by

$$\Upsilon_+(u) = \begin{cases} \displaystyle\int_\Omega G\left(\nabla u^{\frac{1}{q}}\right)dx + \frac{M_{11}}{r_2}\int_{\partial\Omega} |u|^{\frac{r_2}{q}}d\sigma & \text{if } u \ge 0, u^{\frac{1}{q}} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let u_1, u_2 be in the domain of Υ_+, i.e. $u_1, u_2 \in \mathrm{dom}(\Upsilon_+) = \{u \in L^1(\Omega) : \Upsilon_+(u) < +\infty\}$ and let further $u = (tu_1 + (1-t)u_2)^{\frac{1}{q}}$ with $t \in [0,1]$. Applying Lemma 1 in Díaz-Saá [4] there holds

$$\|\nabla u(x)\|_{\mathbb{R}^N} \leq \left(t \left\|\nabla u_1(x)^{\frac{1}{q}}\right\|^q_{\mathbb{R}^N} + (1-t)\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|^q_{\mathbb{R}^N}\right)^{\frac{1}{q}} \quad \text{a.e. in } \Omega.$$

Since G_0 is increasing and thanks to condition H$(a)_1$(iv) it follows

$$\begin{aligned} &G_0\left(\|\nabla u(x)\|_{\mathbb{R}^N}\right) \\ &\leq G_0\left(\left(t \left\|\nabla u_1(x)^{\frac{1}{q}}\right\|^q_{\mathbb{R}^N} + (1-t)\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|^q_{\mathbb{R}^N}\right)^{\frac{1}{q}}\right) \\ &\leq tG_0\left(\left\|\nabla u_1(x)^{\frac{1}{q}}\right\|_{\mathbb{R}^N}\right) + (1-t)G_0\left(\left\|\nabla u_2(x)^{\frac{1}{q}}\right\|_{\mathbb{R}^N}\right) \quad \text{a.e. in } \Omega. \end{aligned}$$

By definition $G(\xi) = G_0(\|\xi\|)$ for all $\xi \in \mathbb{R}^N$, hence

$$G(\nabla u(x)) \leq tG\left(\nabla u_1(x)^{\frac{1}{q}}\right) + (1-t)G\left(\nabla u_2(x)^{\frac{1}{q}}\right) \quad \text{a.e. in } \Omega.$$

Therefore, Υ_+ is convex and due to Fatou's lemma it is also lower semicontinuous.

Now, taking two positive solutions $u, v \in W^{1,p}(\Omega)$ of (3.18) and recalling that $u, v \in \mathrm{int}\left(C^1(\overline{\Omega})_+\right)$ (see the first part of the proof) we have $u, v \in \mathrm{dom}(\Upsilon_+)$. For $h \in C^1(\overline{\Omega})$ and $t \in (0,1)$ sufficiently small we see that $u^q + th, v^q + th \in \mathrm{int}\left(C^1(\overline{\Omega})_+\right)$. Thus, Υ_+ is Gateaux differentiable at u^q and v^q in the direction h. Applying the chain rule and the nonlinear Green's identity (see, for example, Gasiński-Papageorgiou [11, p. 210]) yields

$$\Upsilon_+'\left(u^q\right)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla u)}{u^{q-1}} h dx + \frac{\lambda - \varepsilon_2}{q}\int_{\partial\Omega} h d\sigma, \tag{3.20}$$

$$\Upsilon_+'\left(v^q\right)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla v)}{v^{q-1}} h dx + \frac{\lambda - \varepsilon_2}{q}\int_{\partial\Omega} h d\sigma. \tag{3.21}$$

Since Υ_+' is monotone (follows from the convexity of Υ_+) and thanks to (3.20) as well as (3.21), we obtain

$$\begin{aligned} 0 &\leq \left\langle \Upsilon_+'\left(u^q\right) - \Upsilon_+'\left(v^q\right), u^q - v^q\right\rangle_{L^1(\Omega)} \\ &= \frac{1}{q}\int_\Omega \left(\frac{-\operatorname{div} a(\nabla u)}{u^{q-1}} - \frac{-\operatorname{div} a(\nabla v)}{v^{q-1}}\right)\left(u^q - v^q\right) dx \end{aligned}$$

$$= \frac{1}{q} \int_{\Omega} \left(\frac{-(\chi+\varepsilon_1)u^{p-1} - M_{10}u^{r_1-1}}{u^{q-1}} - \frac{-(\chi+\varepsilon_1)v^{p-1} - M_{10}v^{r_1-1}}{v^{q-1}} \right) \left(u^q - v^q\right) dx$$

$$= \frac{M_{10}}{q} \int_{\Omega} \left(v^{r_1-q} - u^{r_1-q}\right) \left(u^q - v^q\right) dx + \frac{\chi+\varepsilon_1}{q} \int_{\Omega} \left(v^{p-q} - u^{p-q}\right) \left(u^q - v^q\right) dx.$$

Since $s \mapsto s^{r_1-q}$ and $s \mapsto s^{p-q}$ are strictly increasing in $(0, \infty)$ we obtain that $u = v$ and therefore, $u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ is the unique positive solution of (3.18). □

With the aid of these solutions obtained in the last proposition we are now in the position to prove the existence of extremal constant sign solutions of our original problem (1.1) provided $\lambda > 0$ is sufficiently large.

Proposition 3.5 *Let the assumptions in $H(a)_1$ and (H1)–(H8) be satisfied and assume*

$$\lambda > \begin{cases} qc_7\hat{\lambda}_1(q) & \text{if } q < p, \\ 2pc_7\hat{\lambda}_1(p) & \text{if } q = p. \end{cases}$$

Then problem (1.1) has a smallest positive solution $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and a greatest negative solution $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

Proof As mentioned before it is enough to prove the existence of these extremal solutions in the sets $\hat{\mathcal{S}}_+(\lambda) = \mathcal{S}_+(\lambda) \cap [0, \overline{u}] \subseteq \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and $\hat{\mathcal{S}}_-(\lambda) = \mathcal{S}_-(\lambda) \cap [\underline{v}, 0] \subseteq -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

First, we are going to prove that $u_* \leq u$ for all $u \in \hat{S}_+(\lambda)$. For this purpose, let $\hat{v} \in \hat{S}_+(\lambda)$ and define the Carathéodory functions $\zeta^+ : \Omega \times \mathbb{R} \to \mathbb{R}$, $\zeta_\lambda^+ : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ through

$$\zeta^+(x,s) = \begin{cases} 0 & \text{if } s < 0 \\ -\varepsilon_1 s^{p-1} - M_{10}s^{r_1-1} & \text{if } 0 \leq s \leq \hat{v}(x)\ , \\ -\varepsilon_1 \hat{v}(x)^{p-1} - M_{10}\hat{v}(x)^{r_1-1} & \text{if } \hat{v}(x) < s \end{cases} \tag{3.22}$$

and

$$\zeta_\lambda^+(x,s) = \begin{cases} 0 & \text{if } s < 0 \\ (\lambda - \varepsilon_2)s^{q-1} - M_{11}s^{r_2-1} & \text{if } 0 \leq s \leq \hat{v}(x)\ . \\ (\lambda - \varepsilon_2)\hat{v}(x)^{q-1} - M_{11}\hat{v}(x)^{r_2-1} & \text{if } \hat{v}(x) < s \end{cases} \tag{3.23}$$

Moreover, we define the C^1-functional $\Xi_\lambda^+ : W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\Xi_\lambda^+(u) = \int_{\Omega} G(\nabla u)dx + \frac{\chi}{p} \int_{\Omega} |u|^p dx - \int_{\Omega} Z^+(x,u)dx - \int_{\partial\Omega} Z_\lambda^+(x,u)d\sigma$$

with $Z^+(x,s)=\int_0^s \zeta^+(x,t)dt$ and $Z_\lambda^+(x,s)=\int_0^s \zeta_\lambda^+(x,t)dt$. Thanks to the truncations we easily verify that Ξ_λ^+ is coercive and sequentially weakly lower semicontinuous. Hence, there exists the global minimizer of Ξ_λ^+ on $W^{1,p}(\Omega)$, i.e.

$$\Xi_\lambda^+(\hat{u}_*)=\inf\left[\Xi_\lambda^+(u): u\in W^{1,p}(\Omega)\right]. \tag{3.24}$$

As in the proof of Proposition 3.3 we can show that

$$\Xi_\lambda^+\left(\hat{u}_*\right)<0=\Xi_\lambda^+(0),$$

meaning $\hat{u}_*\neq 0$. Moreover, due to (3.24), there holds

$$\begin{aligned}&\int_\Omega \left(a(\nabla\hat{u}_*),\nabla\varphi\right)_{\mathbb{R}^N}dx+\chi\int_\Omega |\hat{u}_*|^{p-2}\hat{u}_*\varphi dx\\&\quad=\int_\Omega \zeta^+\left(x,\hat{u}_*\right)\varphi dx+\int_{\partial\Omega}\zeta_\lambda^+\left(x,\hat{u}_*\right)\varphi d\sigma,\end{aligned} \tag{3.25}$$

for all $\varphi\in W^{1,p}(\Omega)$. Taking $\varphi=-\hat{u}_*^-\in W^{1,p}(\Omega)$ in (3.25) and applying Lemma 2.4(iii) we derive

$$\min\left(\frac{c_2}{p-1},\chi\right)\left(\|\nabla\hat{u}_*^-\|_{p,\Omega}^p+\|\hat{u}_*^-\|_{p,\Omega}^p\right)\le 0,$$

which implies $\hat{u}_*\ge 0$. Since $\hat{v}$ is a positive solution of (1.1) it satisfies

$$\begin{aligned}&\int_\Omega \left(a\left(\nabla\hat{v}\right),\nabla\varphi\right)_{\mathbb{R}^N}dx+\chi\int_\Omega \hat{v}^{p-1}\varphi dx\\&\quad=\int_\Omega\left(-f\left(x,\hat{v}\right)\right)\varphi dx+\int_{\partial\Omega}\left(\lambda\hat{v}^{q-1}-h\left(x,\hat{v}\right)\right)\varphi d\sigma,\end{aligned} \tag{3.26}$$

for all $\varphi\in W^{1,p}(\Omega)$. Choosing $\left(\hat{u}_*-\hat{v}\right)^+\in W^{1,p}(\Omega)$ in (3.25) and (3.26), subtracting (3.26) from (3.25), and making use of (3.17), (3.22) as well as (3.23) we derive

$$\begin{aligned}&\int_\Omega\left(a\left(\nabla\hat{u}_*\right)-a\left(\nabla\hat{v}\right),\nabla\left(\hat{u}_*-\hat{v}\right)^+\right)_{\mathbb{R}^N}dx+\chi\int_\Omega\left((\hat{u}_*)^{p-1}-\hat{v}^{p-1}\right)\left(\hat{u}_*-\hat{v}\right)^+dx\\&\quad=\int_\Omega\left(\zeta^+(x,\hat{u}_*)+f\left(x,\hat{v}\right)\right)\left(\hat{u}_*-\hat{v}\right)^+dx\\&\qquad+\int_{\partial\Omega}\left(\zeta_\lambda^+(x,\hat{u}_*)-\lambda\hat{v}^{q-1}+h\left(x,\hat{v}\right)\right)\left(\hat{u}_*-\hat{v}\right)^+d\sigma\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega} \left(-\varepsilon_1 \hat{v}^{p-1} - M_{10}\hat{v}^{r_1-1} + f\left(x, \hat{v}\right)\right) \left(\hat{u}_* - \hat{v}\right)^+ dx \\
&\quad + \int_{\partial\Omega} \left(\left(\lambda - \varepsilon_2\right) \hat{v}^{q-1} - M_{11}\hat{v}^{r_2-1} - \lambda \hat{v}^{q-1} + h\left(x, \hat{v}\right)\right) \left(\hat{u}_* - \hat{v}\right)^+ d\sigma \\
&\leq 0.
\end{aligned}
$$

This implies for $\hat{u}_* > \hat{v}$, due to Lemma 2.4(i),

$$
\begin{aligned}
0 &\geq \int_{\Omega} \left(a\left(\nabla \hat{u}_*\right) - a\left(\nabla \hat{v}\right), \nabla \left(\hat{u}_* - \hat{v}\right)^+\right)_{\mathbb{R}^N} dx \\
&\quad + \chi \int_{\Omega} \left(\left(\hat{u}_*\right)^{p-1} - \hat{v}^{p-1}\right) \left(\hat{u}_* - \hat{v}\right)^+ dx \\
&> 0,
\end{aligned}
$$

which is a contradiction. Therefore $\hat{u}_* \leq \hat{v}$. To sum up, we have shown that $\hat{u}_* \in \left[0, \hat{v}\right]$ and $\hat{u}_* \not\equiv 0$. Then, by definition of the truncations, we obtain that $\hat{u}_*$ is a positive solution of (3.18) which by Proposition 3.4 implies that $\hat{u}_* = u_* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Hence

$$
u_* \leq u \qquad \text{for all} \quad u \in \hat{\mathcal{S}}_+(\lambda). \tag{3.27}
$$

Now let $\mathcal{C} \subseteq \hat{\mathcal{S}}_+(\lambda)$ be a chain, that means, a totally ordered subset of $\hat{\mathcal{S}}_+(\lambda)$. Then there exists a sequence $(u_n)_{n\geq 1} \subseteq \hat{\mathcal{S}}_+(\lambda)$ (see Dunford-Schwartz [5, p. 336]) such that $\inf \mathcal{C} = \inf_{n\geq 1} u_n$. As u_n is a positive solution of (1.1) we have

$$
\begin{aligned}
&\int_{\Omega} (a(\nabla u_n), \nabla \varphi)_{\mathbb{R}^N} dx \\
&\quad = \int_{\Omega} \left(-\chi u_n^{p-1} - f(x, u_n)\right) \varphi dx + \int_{\partial\Omega} \left(\lambda u_n^{q-1} - h(x, u_n)\right) \varphi d\sigma,
\end{aligned} \tag{3.28}
$$

for all $\varphi \in W^{1,p}(\Omega)$ with $u_* \leq u_n \leq \overline{u}$ for all $n \geq 1$ (see (3.27)). Since f and h are bounded on bounded sets we obtain the boundedness of u_n in $W^{1,p}(\Omega)$. Therefore, we may assume that

$$
u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega), \qquad u_n \to u \text{ in } L^p(\Omega), \qquad u_n \to u \text{ in } L^p(\partial\Omega). \tag{3.29}
$$

Taking $\varphi = u_n - u \in W^{1,p}(\Omega)$ in (3.28) and passing to the limit as $n \to \infty$, one gets, thanks to the convergence properties in (3.29),

$$
\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = \lim_{n\to\infty} \int_{\Omega} (a(\nabla u_n), \nabla(u_n - u))_{\mathbb{R}^N} dx = 0. \tag{3.30}
$$

Since A satisfies the (S_+)-property (see Proposition 2.7), (3.29) and (3.30) imply $u_n \to u$ in $W^{1,p}(\Omega)$. Using this fact we can pass again to the limit in (3.28) which gives

$$\int_{\Omega} (a(\nabla u), \nabla \varphi)_{\mathbb{R}^N} \, dx = \int_{\Omega} (-\chi u^{p-1} - f(x,u))\varphi dx + \int_{\partial\Omega} (\lambda u^{q-1} - h(x,u)\varphi d\sigma$$

with $u_* \leq u \leq \overline{u}$. That means $u \in \hat{\mathcal{S}}_+(\lambda)$ and $u = \inf \mathcal{C}$. Then, the Kuratowski-Zorn Lemma implies that $\hat{\mathcal{S}}_+(\lambda)$ has a minimal element $u_+ \in \hat{\mathcal{S}}_+(\lambda)$ and since $\hat{\mathcal{S}}_+(\lambda)$ is downward directed, we infer that u_+ is the smallest positive solution of (1.1).

The existence of a greatest negative solution $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ of (1.1) can be shown in the same way, working with the set $\hat{\mathcal{S}}_-(\lambda)$ instead of $\hat{\mathcal{S}}_+(\lambda)$. The proof is complete. □

Now, we are going to prove the existence of a nontrivial solution y_0 of (1.1) which turns out to be a sign changing solution.

Proposition 3.6 *If hypotheses H(a)$_1$ and (H1)–(H8) hold and if*

$$\lambda > \begin{cases} qc_7\hat{\lambda}_2(q) & \text{if } q < p \\ 2pc_7\hat{\lambda}_2(p) & \text{if } q = p \end{cases}$$

is satisfied, then problem (1.1) *has a nodal solution* $y_0 \in C^1\left(\overline{\Omega}\right)$.

Proof Recall that $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ are the two extremal constant sign solutions of (1.1) obtained by Proposition 3.5. Let $\vartheta : \Omega \times \mathbb{R} \to \mathbb{R}$, $\vartheta_\lambda : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, be truncation functions defined by

$$\vartheta(x,s) = \begin{cases} -f\left(x, v_-(x)\right) & \text{if } s < v_-(x) \\ -f\left(x, s\right) & \text{if } v_-(x) \leq s \leq u_+(x) \\ -f\left(x, u_+(x)\right) & \text{if } u_+(x) < s \end{cases} \tag{3.31}$$

and

$$\vartheta_\lambda(x,s) = \begin{cases} \lambda \left|v_-(x)\right|^{q-2} v_-(x) - h\left(x, v_-(x)\right) & \text{if } s < v_-(x) \\ \lambda \left|s\right|^{q-2} s - h\left(x, s\right) & \text{if } v_-(x) \leq s \leq u_+(x) \\ \lambda u_+(x)^{q-1} - h\left(x, u_+(x)\right) & \text{if } u_+(x) < s \end{cases}. \tag{3.32}$$

Furthermore, let $\vartheta^{\pm}(x,s)=\vartheta\left(x,\pm s^{\pm}\right)$, $\vartheta_{\lambda}^{\pm}(x,s)=\vartheta_{\lambda}\left(x,\pm s^{\pm}\right)$ and define

$$\Theta(x,s)=\int_0^s\vartheta(x,t)\,dt,\qquad \Theta_{\lambda}(x,s)=\int_0^s\vartheta_{\lambda}(x,t)\,dt,$$
$$\Theta^{\pm}(x,s)=\int_0^s\vartheta^{\pm}(x,t)\,dt,\qquad \Theta_{\lambda}^{\pm}(x,s)=\int_0^s\vartheta_{\lambda}^{\pm}(x,t)\,dt.$$

We consider the C^1-functionals $\Phi_{\lambda},\Phi_{\lambda}^{\pm}:W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\Phi_{\lambda}(u)=\int_{\Omega}G(\nabla u)\,dx+\frac{\chi}{p}\int_{\Omega}|u|^pdx-\int_{\Omega}\Theta(x,u)dx-\int_{\partial\Omega}\Theta_{\lambda}(x,u)d\sigma,$$
$$\Phi_{\lambda}^{\pm}(u)=\int_{\Omega}G(\nabla u)dx+\frac{\chi}{p}\int_{\Omega}|u|^pdx-\int_{\Omega}\Theta^{\pm}(x,u)dx-\int_{\partial\Omega}\Theta_{\lambda}^{\pm}(x,u)d\sigma.$$

First, we will prove that

$$K_{\Phi_{\lambda}}\subseteq[v_-,u_+],\quad K_{\Phi_{\lambda}^{+}}=\{0,u_+\},\quad K_{\Phi_{\lambda}^{-}}=\{v_-,0\}. \tag{3.33}$$

To this end, let $u\in K_{\Phi_{\lambda}}$, that is

$$\begin{aligned}&\int_{\Omega}(a(\nabla u),\nabla\varphi)_{\mathbb{R}^N}\,dx+\chi\int_{\Omega}|u|^{p-2}u\varphi dx\\&\quad=\int_{\Omega}\vartheta(x,u)\varphi dx+\int_{\partial\Omega}\vartheta_{\lambda}(x,u)\varphi d\sigma\quad\text{for all}\quad\varphi\in W^{1,p}(\Omega).\end{aligned} \tag{3.34}$$

Since u_+ is a positive solution of (1.1), we have

$$\begin{aligned}&\int_{\Omega}(a(\nabla u_+),\nabla\varphi)_{\mathbb{R}^N}\,dx+\chi\int_{\Omega}u_+^{p-1}\varphi dx\\&\quad=\int_{\Omega}(-f(x,u_+))\varphi dx+\int_{\partial\Omega}\left(\lambda u_+^{p-1}-h(x,u_+)\right)\varphi d\sigma,\end{aligned} \tag{3.35}$$

for all $\varphi \in W^{1,p}(\Omega)$. Choosing $\varphi = (u - u_+)^+ \in W^{1,p}(\Omega)$ in (3.34) and (3.35) and subtracting (3.35) from (3.34) results in

$$\begin{aligned}
&\int_\Omega \left(a(\nabla u) - a(\nabla u_+), \nabla (u - u_+)^+\right)_{\mathbb{R}^N} dx + \chi \int_\Omega \left(|u|^{p-2}u - u_+^{p-1}\right)(u - u_+)^+ \, dx \\
&\quad = \int_\Omega (\vartheta(x, u) + f(x, u_+))(u - u_+)^+ \, dx \\
&\qquad + \int_{\partial\Omega} \left(\vartheta_\lambda(x, u) - \lambda u_+^{p-1} + h(x, u_+)\right)(u - u_+)^+ \, d\sigma \\
&\quad = 0,
\end{aligned}$$

due to the definition of the truncations in (3.31) and (3.32). Since $a : \mathbb{R}^N \to \mathbb{R}^N$ is strictly monotone (see Lemma 2.4(i)), we derive for $u > u_+$

$$\begin{aligned}
0 &= \int_\Omega \left(a(\nabla u) - a(\nabla u_+), \nabla (u - u_+)^+\right)_{\mathbb{R}^N} + \chi \int_\Omega \left(|u|^{p-2}u - u_+^{p-1}\right)(u - u_+)^+ \, dx \\
&> 0,
\end{aligned}$$

which is a contradiction. This gives $u \le u_+$.

Acting on (3.34) and (3.35) with $\varphi = (v_- - u)^+ \in W^{1,p}(\Omega)$ and subtracting again we obtain $v_- \le u$. Hence

$$K_{\Phi_\lambda} \subseteq [v_-, u_+].$$

Following the same ideas we can prove that

$$K_{\Phi_\lambda^+} \subseteq [0, u_+] \quad \text{and} \quad K_{\Phi_\lambda^-} \subseteq [v_-, 0].$$

By Proposition 3.5 we have that u_+ and v_- are the extremal constant sign solutions of (1.1). Therefore

$$K_{\Phi_\lambda^+} = \{0, u_+\} \quad \text{and} \quad K_{\Phi_\lambda^-} = \{v_-, 0\}.$$

This proves (3.33).

Next, we are going to show that

$$\begin{aligned}
&u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \text{ and } v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right) \text{ are local} \\
&\text{minimizers of } \Phi_\lambda.
\end{aligned} \tag{3.36}$$

We easily verify that the functional Φ_λ^+ is coercive and sequentially weakly lower semicontinuous. Then, by the Weierstrass theorem, there exists $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\Phi_\lambda^+\left(\hat{u}\right) = \inf\left\{\Phi_\lambda^+(u) : u \in W^{1,p}(\Omega)\right\}.$$

Applying the same arguments as in the proof of Proposition 3.3, we can show that $\Phi_\lambda^+\left(\hat{u}\right) < 0 = \Phi_\lambda^+(0)$ which implies $\hat{u} \neq 0$. Hence, because of (3.33), $\hat{u} = u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Since $\Phi_\lambda\big|_{C^1(\overline{\Omega})_+} = \Phi_\lambda^+\big|_{C^1(\overline{\Omega})_+}$ we know that $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ is a local $C^1(\overline{\Omega})$-minimizer of Φ_λ and thanks to Proposition 2.6 it follows that u_+ is a local $W^{1,p}(\Omega)$-minimizer of Φ_λ. The assertion for Φ_λ^- can be shown using similar arguments. This proves (3.36).

Now, we may assume, without loss of generality, that $\Phi_\lambda(v_-) \leq \Phi_\lambda(u_+)$ and that u_+ is an isolated element of K_{Φ_λ}, otherwise we would have a whole sequence of distinct nontrivial solutions of (1.1). Then, we can find a number $\rho \in (0, 1)$ such that $\|v_- - u_+\|_{1,p} > \rho$ and

$$\Phi_\lambda\left(v_-\right) \leq \Phi_\lambda\left(u_+\right) < \inf\left[\Phi_\lambda(u) : \|u - u_+\|_{1,p} = \rho\right] = m_\rho^\lambda. \tag{3.37}$$

From Proposition 2.9 we have that Φ_λ satisfies the PS-condition because it is coercive. This fact along with (3.37) allow us the application of the mountain pass theorem stated in Theorem 2.2 which guarantees the existence of $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_{\Phi_\lambda} \quad \text{and} \quad m_\rho^\lambda \leq \Phi_\lambda(y_0). \tag{3.38}$$

Note that the first assertion in (3.38) combined with (3.33) and the definition of the truncations in (3.31), (3.32) implies that y_0 is a solution of problem (1.1). The second assertion in (3.38) along with (3.37) gives $y_0 \notin \{v_-, u_+\}$ and the nonlinear regularity theory yields that $y_0 \in C^1(\overline{\Omega})$. Since v_- and u_+ are the extremal constant sign solutions of (1.1) we know that $y_0 \in [v_-, u_+]\setminus\{v_-, u_+\}$ has changing sign provided $y_0 \neq 0$.

Moreover, since y_0 is of mountain pass type, we obtain, due to Theorem 2.2,

$$\Phi_\lambda(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \Phi_\lambda(\gamma(t)), \tag{3.39}$$

where $\Gamma = \left\{\gamma \in C\left([0, 1], W^{1,p}(\Omega)\right) : \gamma(0) = v_-, \gamma(1) = u_+\right\}$. In order to complete the proof we have to show that y_0 is unequal zero which is satisfied if there exists a path $\gamma_* \in \Gamma$ such that (see (3.39))

$$\Phi_\lambda(\gamma_*(t)) \neq 0 \quad \text{for all} \quad t \in [0, 1].$$

Taking hypotheses (H3), (H6) into account, for given $\varepsilon_1, \varepsilon_2 > 0$, there exist numbers $\delta_1 = \delta_1(\varepsilon_1), \delta_2 = \delta_2(\varepsilon_2) > 0$ such that

$$|f(x, s)| \leq \varepsilon_1 |s|^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \leq \delta_1,$$
$$|h(x, s)| \leq \varepsilon_2 |s|^{q-1} \quad \text{for a.a. } x \in \partial\Omega \text{ and all } |s| \leq \delta_2,$$

which implies that

$$\begin{aligned} F(x,s) &\leq \frac{\varepsilon_1}{p}|s|^p \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \leq \delta_1, \\ H(x,s) &\leq \frac{\varepsilon_2}{q}|s|^q \quad \text{for a.a. } x \in \partial\Omega \text{ and all } |s| \leq \delta_2. \end{aligned} \tag{3.40}$$

Let $S_q = W^{1,q}(\Omega) \cap \partial B_1^{q,\partial\Omega}$ and $S_q^c = S_q \cap C^1(\overline{\Omega})$ be equipped with the relative $W^{1,p}(\Omega)$-topology and the relative $C^1(\overline{\Omega})$-topology, respectively. Recall that $\partial B_1^{q,\partial\Omega} = \left\{u \in L^q(\partial\Omega) : \|u\|_{q,\partial\Omega} = 1\right\}$ and

$$\hat{\Gamma}(q) = \left\{\hat{\gamma} \in C\left([-1,1], S_q\right) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\right\}.$$

Moreover, we consider the set of continuous paths

$$\hat{\Gamma}_c(q) = \left\{\hat{\gamma} \in C\left([-1,1], S_q^c\right) : \hat{\gamma}(-1) = -\hat{u}_1(q), \hat{\gamma}(1) = \hat{u}_1(q)\right\}.$$

Let $\delta := \min\{\delta_1, \delta_2\}$. From the variational characterization of the second eigenvalue $\hat{\lambda}_2(q)$ (see Proposition 2.8), we find $\hat{\gamma} \in \hat{\Gamma}(q)$ such that

$$\max_{-1 \leq t \leq 1} \|\hat{\gamma}(t)\|_{1,q}^q < \hat{\lambda}_2(q) + \frac{\delta}{2}. \tag{3.41}$$

It is well known that S_q^c is dense in S_q. This implies the density of $\hat{\Gamma}_c(q)$ in $\hat{\Gamma}(q)$ (see, for example, Winkert [22, Proof of Theorem 3.1.16]). Therefore, for a given $\varepsilon > 0$, there exists $\hat{\gamma}_0 \in \hat{\Gamma}_c(q)$ such that

$$\max_{-1 \leq t \leq 1} \|\hat{\gamma}(t) - \hat{\gamma}_0(t)\|_{1,q} < \varepsilon. \tag{3.42}$$

Selecting $\varepsilon \in \left(0, \left(\hat{\lambda}_2(q) + \delta\right)^{\frac{1}{q}} - \left(\hat{\lambda}_2(q) + \frac{\delta}{2}\right)^{\frac{1}{q}}\right)$ we derive from (3.41) and (3.42)

$$\begin{aligned} \|\hat{\gamma}_0(t)\|_{1,q} &\leq \|\hat{\gamma}_0(t) - \hat{\gamma}(t)\|_{1,q} + \|\hat{\gamma}(t)\|_{1,q} \\ &< \varepsilon + \left(\hat{\lambda}_2(q) + \frac{\delta}{2}\right)^{\frac{1}{q}} \\ &< \left(\hat{\lambda}_2(q) + \delta\right)^{\frac{1}{q}} \quad \text{for all } \quad t \in [-1,1], \end{aligned}$$

which results in

$$\max_{-1 \leq t \leq 1} \|\hat{\gamma}_0(t)\|_{1,q}^q < \hat{\lambda}_2(q) + \delta. \tag{3.43}$$

Recall that $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ and $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$. Then, since $\hat{\gamma}_0\left([-1,1]\right) \subseteq C^1(\overline{\Omega})$ is compact, there exists a number $\xi \in (0,1)$ such that

$$|\xi u(x)| \leq \delta \quad \text{for all} \quad x \in \overline{\Omega}, \text{ for all} \quad u \in \hat{\gamma}_0\left([-1,1]\right), \tag{3.44}$$

and

$$\xi u \in [v_-, u_+] \quad \text{for all} \quad u \in \hat{\gamma}_0\left([-1,1]\right).$$

Due to (3.1) along with $\|u\|_{q,\partial\Omega} = 1$, (3.31), (3.32), (3.40), (3.43), (3.44), and $u \in \hat{\gamma}_0\left([-1,1]\right)$, it follows

$$\begin{aligned}
\Phi_\lambda(\xi u) &= \int_\Omega G(\nabla(\xi u))dx + \frac{\chi}{p}\int_\Omega |\xi u|^p dx - \int_\Omega \Theta(x,\xi u)dx - \int_{\partial\Omega} \Theta_\lambda(x,\xi u)d\sigma \\
&\leq c_7\left(\xi^q\|\nabla u\|_{q,\Omega}^q + \xi^p\|\nabla u\|_{p,\Omega}^p\right) + \frac{\chi\xi^p}{p}\|u\|_{p,\Omega}^p \\
&\quad + \int_\Omega F(x,\xi u)dx - \frac{\lambda\xi^q}{q} + \int_{\partial\Omega} H(x,\xi u)d\sigma \\
&\leq c_7\left(\xi^q\|\nabla u\|_{q,\Omega}^q + \xi^q\|u\|_{q,\Omega}^q + \xi^p\|\nabla u\|_{p,\Omega}^p\right) - \xi^q c_7\|u\|_{q,\Omega}^q \\
&\quad + \frac{\chi\xi^p}{p}\|u\|_{p,\Omega}^p + \frac{\varepsilon_1\xi^p}{p}\|u\|_{p,\Omega}^p - \frac{\lambda\xi^q}{q} + \frac{\varepsilon_2}{q}\xi^q \\
&\leq \xi^q\left[\frac{qc_7\left(\hat{\lambda}_2(q)+\delta\right) - \lambda + \varepsilon_2}{q}\right] - \xi^q c_7\|u\|_{q,\Omega}^q \\
&\quad + \xi^p\left(c_7\|\nabla u\|_{p,\Omega}^p + \frac{\chi+\varepsilon_1}{p}\|u\|_{p,\Omega}^p\right).
\end{aligned} \tag{3.45}$$

Furthermore, since $\hat{\gamma}_0\left([-1,1]\right) \subseteq C^1(\overline{\Omega})$ is compact, we find a number $\xi^* > 0$ such that

$$\|u\|_{1,p}^p \leq \xi^* \quad \text{for all} \quad u \in \hat{\gamma}_0([-1,1]). \tag{3.46}$$

First, suppose that $q < p$. We choose $\varepsilon_1 \in (0,\infty)$, $\varepsilon_2 > 0$ and $\delta \in (0,\varepsilon_2)$ such that $qc_7\delta + \varepsilon_2 < \lambda - qc_7\hat{\lambda}_2(q)$. Taking (3.46) into account, (3.45) becomes

$$\Phi_\lambda(\xi u) \leq -\xi^q M_{12} + \xi^p M_{13} \quad \text{for some} \quad M_{12}, M_{13} > 0.$$

Since $q < p$, by choosing $\xi \in (0,1)$ small enough, we obtain

$$\Phi_\lambda(\xi u) < 0 \quad \text{for all} \quad u \in \hat{\gamma}_0([-1,1]). \tag{3.47}$$

If $q = p$, by applying (3.46) and again (3.43), estimate (3.45) becomes

$$\begin{aligned}\Phi_\lambda(\xi u) &\leq \xi^p \left[\frac{pc_7\left(\hat{\lambda}_2(p)+\delta\right)-\lambda+\varepsilon_2}{p}\right] - \xi^p c_7 \|u\|_{p,\Omega}^p \\ &\quad + \xi^p \left(c_7\left(\|\nabla u\|_{p,\Omega}^p + \|u\|_{p,\Omega}^p\right) - c_7\|u\|_{p,\Omega}^p + \frac{\chi+\varepsilon_1}{p}\|u\|_{p,\Omega}^p\right) \\ &\leq \xi^p \left[\frac{2pc_7\left(\hat{\lambda}_2(p)+\delta\right)-\lambda+\varepsilon_2}{p}\right] + \xi^p \frac{\chi+\varepsilon_1-2pc_7}{p}\xi^*.\end{aligned} \tag{3.48}$$

If $2pc_7 > \chi$, we choose $0 < \varepsilon_1 < 2pc_7 - \chi$ as well as $\varepsilon_2 > 0$ and $\delta \in (0, \varepsilon_2)$ such that $pc_7\delta + \varepsilon_2 < \lambda - pc_7\hat{\lambda}_2(p)$ which proves (3.47). If $2pc_7 = \chi$, then we select again $\varepsilon_2 > 0$ and $\delta \in (0, \varepsilon_2)$ such that $pc_7\delta + \varepsilon_2 < \lambda - pc_7\hat{\lambda}_2(p)$ for which (3.48) results in

$$\Phi_\lambda(\xi u) \leq \xi^p \left[-M_{14} + \frac{\varepsilon_1}{p}\xi^*\right] \tag{3.49}$$

with some $M_{14} > 0$. Choosing $0 < \varepsilon_1 < \frac{pM_{14}}{\xi^*}$ proves (3.47) in this case, too.
Now, we set $\gamma_0 = \xi\hat{\gamma}_0$ which is a continuous path in $W^{1,p}(\Omega)$ connecting $-\xi\hat{u}_1(q)$ and $\xi\hat{u}_1(q)$ and which fulfills

$$\Phi_\lambda\big|_{\gamma_0} < 0. \tag{3.50}$$

Recall that, due to (3.33), $K_{\Phi_\lambda^+} = \{0, u_+\}$. Moreover, the proof of (3.36) shows that

$$\Phi_\lambda^+(u_+) = \inf_{u\in W^{1,p}(\Omega)} \Phi_\lambda^+(u) < 0 = \Phi_\lambda^+(0). \tag{3.51}$$

Now we may apply the second deformation theorem stated in Theorem 2.3 with $\varphi = \Phi_\lambda^+$, $a = \Phi_\lambda^+(u_+) < 0 = \Phi_\lambda^+(0) = b$ to find a continuous map $\hat{h} : [0,1] \times \left(\left(\Phi_\lambda^+\right)^0 \setminus\{0\}\right) \to \left(\Phi_\lambda^+\right)^0$ such that, because of (3.51) and (3.33),

$$\hat{h}\left(1, \left(\Phi_\lambda^+\right)^0 \setminus\{0\}\right) = \{u_+\} \tag{3.52}$$

and

$$\Phi_\lambda^+(\hat{h}(t,u)) \leq \Phi_\lambda^+(u) \quad \text{for all} \quad t \in [0,1] \quad \text{and all} \quad u \in \left(\Phi_\lambda^+\right)^0 \setminus\{0\}. \tag{3.53}$$

Defining $\gamma_+(t) := \left(\hat{h}(t, \xi\hat{u}_1(q))\right)^+$ for all $t \in [0, 1]$, it is clear that γ_+ is a continuous path in $W^{1,p}(\Omega)$ satisfying

$$\gamma_+(0) = \left(\hat{h}(0, \xi\hat{u}_1(q))\right)^+ = \xi\hat{u}_1(q)$$

and, due to (3.52),

$$\gamma_+(1) = \left(\hat{h}(1, \xi\hat{u}_1(q)\right)^+ = u_+.$$

In addition, thanks to (3.53) and (3.50), one gets

$$\Phi_\lambda^+(\gamma_+(t)) \leq \Phi_\lambda^+(\xi\hat{u}_1(q)) < 0 \quad \text{for all} \quad t \in [0, 1]$$

implying $\Phi_\lambda^+\big|_{\gamma_+} < 0$. Moreover, since

$$\Phi_\lambda\big|_{W_+^p} = \Phi_\lambda^+\big|_{W_+^p} \quad \text{and} \quad \operatorname{im}\gamma_+ \subseteq W_+^p,$$

with $W_+^p = \{u \in W^{1,p}(\Omega) : u(x) \geq 0 \text{ a.e. in } \Omega\}$, we have

$$\Phi_\lambda\big|_{\gamma_+} < 0. \tag{3.54}$$

Following the same ideas we can construct a continuous path γ_- in $W^{1,p}(\Omega)$ which joins v_- and $-\xi\hat{u}_1(q)$ satisfying

$$\Phi_\lambda\big|_{\gamma_-} < 0. \tag{3.55}$$

The union of the curves γ_-, γ_0, and γ_+ forms a continuous path $\gamma_* \in \Gamma$ such that, because of (3.50), (3.54), and (3.55),

$$\Phi_\lambda\big|_{\gamma_*} < 0.$$

This implies that $y_0 \in C^1(\overline{\Omega}) \cap [v_-, u_+]$ is a nodal solution of (1.1). □

Combining the results in Propositions 3.3, 3.5, and 3.6 we have the following multiplicity result.

Theorem 3.7 *Let hypotheses H(a)$_1$ and (H1)–(H8) be satisfied and assume*

$$\lambda > \begin{cases} qc_7\hat{\lambda}_2(q) & \text{if } q < p, \\ 2pc_7\hat{\lambda}_2(p) & \text{if } q = p. \end{cases}$$

Then problem (1.1) *has at least three nontrivial solutions* $u_0 \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$, $v_0 \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ *and a nodal solution* $y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega})$. *Additionally, (1.1) has* *a smallest nontrivial positive solution* $u_+ \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right)$ *and a greatest nontrivial negative solution* $v_- \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right)$.

Remark 3.8 As mentioned in the Introduction recall that the results in Theorem 3.7 recover those ones obtained in Winkert [23]. Indeed, if $q = p$ and $a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\xi$ for all $\xi \in \mathbb{R}^N$ is the p-Laplacian, then $c_7 = \frac{1}{2p}$ and $2pc_7\hat{\lambda}_2(p) = \hat{\lambda}_2(p)$ being the second eigenvalue of the p-Laplacian with Steklov boundary condition. In this case problem (1.1) becomes

$$\begin{aligned} -\Delta_p u &= -\chi|u|^{p-2}u - f(x,u) && \text{in} \quad \Omega, \\ \|\nabla u\|_{\mathbb{R}^N}^{p-2}\frac{\partial u}{\partial n} &= \lambda|u|^{p-2}u - h(x,u) && \text{on} \quad \partial\Omega. \end{aligned}$$

with

$$0 < \chi \leq 1.$$

In contrast to [23] we have on the one hand a more general operator being possibly nonhomogeneous and on the other hand we do not need a sign-changing condition on f near the origin.

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