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Existence of solutions for double phase obstacle problems with multivalued convection term

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A B S T R A C T

The main goal of this paper is the study of an elliptic obstacle problem with a double phase phenomena and a multivalued reaction term which also depends on the gradient of the solution. Such term is called multivalued convection term. Under quite general assumptions on the data, we prove that the set of weak solutions to our problem is nonempty, bounded and closed. Our proof is based on a surjectivity theorem for multivalued mappings generated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping.
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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $1 < p < q < N$. We study the following double phase problem with a multivalued convection term and obstacle effect

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &\in f(x,u,\nabla u) && \text{in } \Omega,\\ u(x) &\le \Phi(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

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where $\mu\colon \overline{\Omega} \to [0,\infty)$ is Lipschitz continuous, $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ is a multivalued function depending on the gradient of the solution and $\Phi\colon \Omega \to \mathbb{R}$ is a given function. The precise conditions on the data will be presented in Section 3.

The novelty of our work is the fact that we combine several different phenomena in one problem. To be more precise problem (1.1) contains

(1) a double phase operator;
(2) a multivalued convection term;
(3) an obstacle restriction.

To the best of our knowledge, this is the first work which combines all these phenomena in one problem. We are going to prove that problem (1.1) has at least one solution. The proof is based on a surjectivity result of Le [19] for multivalued mappings generated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping.

Since (1.1) is an obstacle problem, the solutions of (1.1) are supposed to be in the set

$$\left\{u \in W_0^{1,\mathcal{H}}(\Omega) \;\middle|\; u(x) \le \Phi(x) \text{ for a.a. } x \in \Omega\right\}$$

with a given obstacle $\Phi\colon \Omega \to \overline{\mathbb{R}}_+ = [0,\infty]$ where $W^{1,\mathcal{H}}(\Omega)$ denotes the Sobolev-Musielak-Orlicz space, see Section 2 for its definition. When $\Phi \equiv +\infty$, problem (1.1) becomes the following double phase problem with multivalued convection term

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &\in f(x,u,\nabla u) \quad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

In addition, when f is a single-valued function, the above problem reduces to

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &= f(x,u,\nabla u) \quad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

which was recently studied by Gasiński-Winkert in [17].

Problems of type (1.2) are driven by the so-called double phase operator whose name comes from the fact that its behavior depends on the points where μ vanishes or not. Such problems go back to Zhikov who introduced such classes of operators to describe models of strongly anisotropic materials by treating the functional

$$\omega \mapsto \int \left(|\nabla \omega|^p + \mu(x)|\nabla \omega|^q\right)\, dx, \tag{1.3}$$

see [30], [31], [32] and the monograph of Zhikov-Kozlov-Oleinik [33]. Integral functionals of the form (1.3) have been studied by several authors concerning regularity results and non-standard growth. We refer to Baroni-Colombo-Mingione [3], [4], [5], Baroni-Kussi-Mingione [6], Colombo-Mingione [10], [11], Cupini-Marcellini-Mascolo [12] and Marcellini [21], [22] and the references therein.

Existence results for problems like (1.1) in the case of single-valued equations without convection term have been obtained by several authors, see, for example, Colasuonno-Squassina [9], Gasiński-Papageorgiou [15, Proposition 3.4], Gasiński-Winkert [16], Liu-Dai [20], Perera-Squassina [27] and problems with other general differential operator and a convection term by Gasiński-Papageorgiou [14].

Works which are closely related to ours dealing with certain types of double phase problems can be found in Bahrouni-Rădulescu-Repovš [1], [2], Cencelj-Rădulescu-Repovš [8], Papageorgiou-Rădulescu-Repovš [25], [24], Rădulescu [28] Zhang-Rădulescu [29] and the references therein.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ and let $1 \leq r \leq \infty$. We denote by $L^r(\Omega) := L^r(\Omega; \mathbb{R})$ and $L^r(\Omega; \mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norms

$$\|u\|_r := \left(\int_\Omega |u(x)|^r \, dx \right)^{\frac{1}{r}} \quad \text{for all } u \in L^r(\Omega),$$

and

$$\|w\|_{r,N} := \left(\int_\Omega \|w(x)\|_{\mathbb{R}^N}^r \, dx \right)^{\frac{1}{r}} \quad \text{for all } w \in L^r(\Omega; \mathbb{R}^N),$$

respectively. In what follows, for simplicity, the norms of $L^r(\Omega; \mathbb{R})$ and $L^r(\Omega; \mathbb{R}^N)$ are both denoted $\|\cdot\|_r$, even if we do not mention it explicitly. Moreover, $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ stand for the Sobolev spaces endowed with the norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0}$, respectively. For any $1 < r < \infty$ we denote by r' the conjugate of r, that is, $\frac{1}{r} + \frac{1}{r'} = 1$.

In the entire paper we suppose the following condition:

H(μ): $\mu \colon \overline{\Omega} \to \mathbb{R}_+ = [0, \infty)$ is Lipschitz continuous and $1 < p < q < N$ are chosen such that

$$\frac{q}{p} < 1 + \frac{1}{N}.$$

We consider the function $\mathcal{H} \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\mathcal{H}(x,t) = t^p + \mu(x) t^q \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}_+.$$

Based on the definition of $\mathcal{H}$ we are able to introduce the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ given by

$$L^{\mathcal{H}}(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|) \, dx < +\infty \right\}$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf \left\{ \tau > 0 \;\middle|\; \rho_{\mathcal{H}} \left(\frac{u}{\tau} \right) \leq 1 \right\}.$$

We know that $L^{\mathcal{H}}(\Omega)$ turns out to be uniformly convex and so it is a reflexive Banach space. In addition, we introduce the seminormed function space

$$L^q_\mu(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \mu(x) |u|^q \, dx < +\infty \right\}$$

which is equipped with the seminorm $\|\cdot\|_{q,\mu}$ given by

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x) |u|^q \, dx \right)^{\frac{1}{q}}.$$

It is known that the embeddings

$$L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_\mu(\Omega)$$

are continuous, see Colasuonno-Squassina [9, Proposition 2.15 (i), (iv) and (v)]. Taking into account these embeddings we have the inequalities

$$\min\left\{\|u\|^p_{\mathcal{H}}, \|u\|^q_{\mathcal{H}}\right\} \le \|u\|^p_p + \|u\|^q_{q,\mu} \le \max\left\{\|u\|^p_{\mathcal{H}}, \|u\|^q_{\mathcal{H}}\right\} \tag{2.1}$$

for all $u \in L^{\mathcal{H}}(\Omega)$.

By $W^{1,\mathcal{H}}(\Omega)$ we denote the corresponding Sobolev space which is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \||\nabla u|\|_{\mathcal{H}}$.

By $W^{1,\mathcal{H}}_0(\Omega)$ we denote the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$, that is,

$$W^{1,\mathcal{H}}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{W^{1,\mathcal{H}}(\Omega)}$$

Besides, from condition H(μ) and Colasuonno-Squassina [9, Proposition 2.18] we can see that

$$\|u\|_{1,\mathcal{H},0} = \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W^{1,\mathcal{H}}_0(\Omega),$$

is an equivalent norm on $W^{1,\mathcal{H}}_0(\Omega)$. Now we are able to rewrite (2.1) for the space $W^{1,\mathcal{H}}_0(\Omega)$ in the form

$$\min\left\{\|u\|^p_{1,\mathcal{H},0}, \|u\|^q_{1,\mathcal{H},0}\right\} \le \|\nabla u\|^p_p + \|\nabla u\|^q_{q,\mu} \le \max\left\{\|u\|^p_{1,\mathcal{H},0}, \|u\|^q_{1,\mathcal{H},0}\right\} \tag{2.2}$$

for all $u \in W^{1,\mathcal{H}}_0(\Omega)$. Since both spaces $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ are uniformly convex, we know that they are reflexive Banach spaces.

Furthermore, we have the following compact embedding

$$W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow L^r(\Omega) \tag{2.3}$$

for each $1 < r < p^*$, where p^* is the critical exponent to p given by

$$p^* := \frac{Np}{N-p}, \tag{2.4}$$

see Colasuonno-Squassina [9, Proposition 2.15].

Let us now consider the eigenvalue problem for the r-Laplacian with homogeneous Dirichlet boundary condition and $1 < r < \infty$ which is defined by

$$\begin{aligned} -\Delta_r u &= \lambda |u|^{r-2} u \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

A number $\lambda \in \mathbb{R}$ is an eigenvalue of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$ if problem (2.5) has a nontrivial solution $u \in W_0^{1,r}(\Omega)$ which is called an eigenfunction corresponding to the eigenvalue λ. We denote by σ_r the set of eigenvalues of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$. From Lê [18] we know that the set σ_r has a smallest element $\lambda_{1,r}$ which is positive, isolated, simple and it can be variationally characterized through

$$\lambda_{1,r} = \inf\left\{\frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0\right\}.$$

Let $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ be the operator defined by

$$\langle A(u), v\rangle_{\mathcal{H}} := \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v\,dx, \tag{2.6}$$

for $u, v \in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. The properties of the operator $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ are summarized in the following proposition, see Liu-Dai [20].

Proposition 2.1. *The operator A defined by* (2.6) *is bounded, continuous, monotone (hence maximal monotone) and of type* (S_+).

Next, we recall the notions of pseudomonotonicity and generalized pseudomonotonicity for multivalued operators (see Gasiński-Papageorgiou [13, Definition 1.4.8]).

Definition 2.2. Let X be a real reflexive Banach space. The operator $A\colon X \to 2^{X^*}$ is called

(a) pseudomonotone if the following conditions hold:
  (i) the set $A(u)$ is nonempty, bounded, closed and convex for all $u \in X$.
  (ii) A is upper semicontinuous from each finite-dimensional subspace of X to the weak topology on X^*.
  (iii) if $\{u_n\} \subset X$ with $u_n \rightharpoonup u$ in X and if $u_n^* \in A(u_n)$ is such that

$$\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$

  then to each element $v \in X$, exists $u^*(v) \in A(u)$ with

$$\langle u^*(v), u - v\rangle_{X^*\times X} \le \liminf_{n\to\infty}\langle u_n^*, u_n - v\rangle_{X^*\times X}.$$

(b) generalized pseudomonotone if the following holds: Let $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ with $u_n^* \in A(u_n)$ be such that $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^*. If

$$\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$

then the element u^* lies in $A(u)$ and

$$\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}.$$

It is not difficult to see that every pseudomonotone operator is generalized pseudomonotone, see, for example, Carl-Le-Motreanu [7, Proposition 2.122] or Gasiński-Papageorgiou [13, Proposition 1.4.11]. However, under the additional assumption of boundedness, we obtain the converse statement, see, for example, Carl-Le-Motreanu [7, Proposition 2.123] or Gasiński-Papageorgiou [13, Proposition 1.4.12].

Proposition 2.3. *Let X be a real reflexive Banach space and assume that $A\colon X \to 2^{X^*}$ satisfies the following conditions:*

(i) *For each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of X^*.*
(ii) *$A\colon X \to 2^{X^*}$ is bounded.*
(iii) *If $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* with $u_n^* \in A(u_n)$ and if*

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0,$$

then $u^* \in A(u)$ and

$$\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}.$$

Then the operator $A\colon X \to 2^{X^*}$ is pseudomonotone.

Furthermore, we will state the following surjectivity theorem for multivalued mappings which is formulated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping. The following theorem was proved in Le [19, Theorem 2.2]. We use the notation $B_R(0) := \{u \in X \ : \ \|u\|_X < R\}$.

Theorem 2.4. *Let X be a real reflexive Banach space, let $F\colon D(F) \subset X \to 2^{X^*}$ be a maximal monotone operator, let $G\colon D(G) = X \to 2^{X^*}$ be a bounded multivalued pseudomonotone operator and let $L \in X^*$. Assume that there exist $u_0 \in X$ and $R \ge \|u_0\|_X$ such that $D(F) \cap B_R(0) \neq \emptyset$ and*

$$\langle \xi + \eta - L, u - u_0\rangle_{X^*\times X} > 0$$

for all $u \in D(F)$ with $\|u\|_X = R$, for all $\xi \in F(u)$ and for all $\eta \in G(u)$. Then the inclusion

$$F(u) + G(u) \ni L$$

has a solution in $D(F)$.

3. Main results

We assume the following hypotheses on the multivalued nonlinearity $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$.

H(f): The multivalued convection mapping $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to 2^{\mathbb{R}}$ has nonempty, compact and convex values such that

(i) the multivalued mapping $x \mapsto f(x,s,\xi)$ has a measurable selection for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$;
(ii) the multivalued mapping $(s,\xi) \mapsto f(x,s,\xi)$ is upper semicontinuous;
(iii) there exists $\alpha \in L^{\frac{q_1}{q_1-1}}(\Omega)$ and $a_1, a_2 \ge 0$ such that

$$|\eta| \le a_1|\xi|^{p\frac{q_1-1}{q_1}} + a_2|s|^{q_1-1} + \alpha(x)$$

for all $\eta \in f(x,s,\xi)$, for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where $1 < q_1 < p^*$ with the critical exponent p^* given in (2.4);
(iv) there exist $w \in L^1_+(\Omega)$ and $b_1, b_2 \ge 0$ such that

$$b_1 + b_2\lambda_{1,p}^{-1} < 1,$$

and

$$\eta s \leq b_1|\xi|^p + b_2|s|^p + w(x)$$

for all $\eta \in f(x,s,\xi)$, for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where $\lambda_{1,p}$ is the first eigenvalue of the Dirichlet eigenvalue problem for the p-Laplacian, see (2.5).

Let K be a subset of $W_0^{1,\mathcal{H}}(\Omega)$ defined by

$$K := \left\{ u \in W_0^{1,\mathcal{H}}(\Omega) \;\middle|\; u(x) \leq \Phi(x) \text{ for a.a. } x \in \Omega \right\}, \tag{3.1}$$

where Φ is a function such that

$$\Phi\colon \Omega \to [0,+\infty]. \tag{3.2}$$

It is obvious that the set K is a nonempty, closed and convex subset of $W_0^{1,\mathcal{H}}(\Omega)$.

Remark 3.1. *From assumption* (3.2) *we see that* $0 \in K$*.*

The weak solutions for problem (1.1) are understood in the following sense.

Definition 3.2. We say that $u \in K$ is a weak solution of problem (1.1) if there exists $\eta \in L^{\frac{q_1}{q_1-1}}(\Omega)$ such that $\eta(x) \in f(x,u(x),\nabla u(x))$ for a.a. $x \in \Omega$ and

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla(v-u) + \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla(v-u)\right) dx = \int_\Omega \eta(x)(v-u)\,dx$$

for all $v \in K$, where K is given by (3.1).

The main result of this paper is stated as the next theorem.

Theorem 3.3. *Assume that $H(\mu)$ and $H(f)$ hold. Then the set of solutions of problem (1.1), denoted by $\mathcal{S}$, is nonempty, bounded and closed.*

Proof. We first prove that problem (1.1) has at least one solution.

Let $i\colon W_0^{1,\mathcal{H}}(\Omega) \to L^{q_1}(\Omega)$ be the embedding operator from $W_0^{1,\mathcal{H}}(\Omega)$ to $L^{q_1}(\Omega)$ with its adjoint operator $i^*\colon L^{q_1'}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$. Since $1 < q_1 < p^*$ the embedding operator i is compact and so i^* as well. However, from hypotheses H(f)(i) and (iii), we can use the same process as the proof of Papageorgiou-Vetro-Vetro [26, Proposition 3] to see that the Nemytskij operator $\widetilde{N}_f\colon W_0^{1,\mathcal{H}}(\Omega) \subset L^{q_1}(\Omega) \to 2^{L^{q_1'}(\Omega)}$ associated to the multivalued mapping f given by

$$\widetilde{N}_f(u) := \left\{ \eta \in L^{q_1'}(\Omega) \;\middle|\; \eta(x) \in f(x,u(x),\nabla u(x)) \text{ for a.a. } x \in \Omega \right\}$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$ is well-defined.

Set $N_f := i^* \circ \widetilde{N}_f\colon W_0^{1,\mathcal{H}}(\Omega) \to 2^{W_0^{1,\mathcal{H}}(\Omega)^*}$. Also, let us consider the indicator function $I_K\colon W_0^{1,\mathcal{H}}(\Omega) \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ of K defined by

$$I_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Under the definitions above, it is not difficult to see that $u \in K$ is a weak solution of problem (1.1), see Definition 3.2, if and only if u solves the following inequality:

Find $u \in K$ and $\eta \in N_f(u)$ such that

$$\langle A(u) - \eta, v - u\rangle_{\mathcal{H}} + I_K(v) - I_K(u) \geq 0 \tag{3.3}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$ where $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ is given in (2.6).

Consider the multivalued operator $\mathcal{A}\colon W_0^{1,\mathcal{H}}(\Omega) \to 2^{W_0^{1,\mathcal{H}}(\Omega)^*}$ defined by

$$\mathcal{A}(u) = A(u) - N_f(u) \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega).$$

Then, using a standard procedure, we can reformulate problem (3.3) to the following inclusion problem: Find $u \in K$ such that

$$\mathcal{A}(u) + \partial I_K(u) \ni 0, \tag{3.4}$$

where the notation ∂I_K stands for the subdifferential of I_K in the sense of convex analysis.

We are going to apply the surjectivity result for multivalued pseudomonotone operators, see Theorem 2.4. To this end, for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ and $\eta \in N_f(u)$, by condition H(f)(iii), we obtain

$$\begin{aligned}
\|\eta\|^{q_1'}_{W_0^{1,\mathcal{H}}(\Omega)^*} &\leq \|i^*\|^{q_1'}\|\xi\|^{q_1'}_{L^{q_1'}(\Omega)} = \|i^*\|^{q_1'} \int_\Omega |\xi(x)|^{q_1'}\,dx \\
&\leq C_0 \int_\Omega \left(a_1|\nabla u(x)|^{p\frac{q_1-1}{q_1}} + a_2|u(x)|^{q_1-1} + \alpha(x)\right)^{q_1'} dx \\
&\leq C_1\left(\|\nabla u\|_p^p + \|u\|_{q_1}^{q_1} + \|\alpha\|_{q_1'}^{q_1'}\right)
\end{aligned} \tag{3.5}$$

for some $C_0, C_1 > 0$, where $\xi \in \widetilde{N}_f(u)$ is such that $\eta = i^*\xi$. This combined with $W_0^{1,\mathcal{H}}(\Omega) \subset W_0^{1,p}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega) \subset L^{q_1}(\Omega)$, $1 < q_1 < p^*$ and Proposition 2.1 implies that $\mathcal{A}\colon W_0^{1,\mathcal{H}}(\Omega) \to 2^{W_0^{1,\mathcal{H}}(\Omega)^*}$ is a bounded mapping.

We claim that $\mathcal{A}$ is pseudomonotone. In order to prove this, we are going to apply Proposition 2.3. Indeed, by hypotheses H(f) we know that $\mathcal{A}$ has nonempty, closed and convex values. Moreover, as we just showed, $\mathcal{A}$ is a bounded mapping. So, it is enough to verify that $\mathcal{A}$ is a generalized pseudomonotone operator.

Let $\{u_n\} \subset W_0^{1,\mathcal{H}}(\Omega)$, $\{u_n^*\} \subset W_0^{1,\mathcal{H}}(\Omega)^*$ and $u \in W_0^{1,\mathcal{H}}(\Omega)$ be such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,\mathcal{H}}(\Omega), \quad u_n^* \rightharpoonup u^* \text{ in } W_0^{1,\mathcal{H}}(\Omega)^*, \tag{3.6}$$

$$u_n^* \in \mathcal{A}(u_n) \quad \text{for all } n \in \mathbb{N},$$

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{\mathcal{H}} \leq 0. \tag{3.7}$$

So, for each $n \in \mathbb{N}$, we are able to find an element $\xi_n \in \widetilde{N}_f(u_n)$ such that $u_n^* = A(u_n) - i^*\xi_n$. From the fact that the embedding from $W_0^{1,\mathcal{H}}(\Omega)$ to $L^{q_1}(\Omega)$ is compact, see (2.3), we have $u_n \to u$ in $L^{q_1}(\Omega)$. Moreover, from (3.5), we see that the sequence $\{\xi_n\}$ is bounded in $L^{q_1'}(\Omega)$. So, (3.7) leads to

$$\begin{aligned}\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle_{\mathcal{H}} &\le \limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle_{\mathcal{H}} - \limsup_{n\to\infty}\langle \xi_n,u_n-u\rangle_{L^{q_1}(\Omega)}\\ &\le \limsup_{n\to\infty}\langle A(u_n)-i^*\xi_n,u_n-u\rangle_{\mathcal{H}}\\ &= \limsup_{n\to\infty}\langle u_n^*,u_n-u\rangle_{\mathcal{H}} \le 0.\end{aligned}$$

This fact along with (3.6) and the (S$_+$)-property of A, see Proposition 2.1, implies that $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. This yields

$$\langle u_n^*,u_n\rangle_{\mathcal{H}} \to \langle u^*,u\rangle_{\mathcal{H}} \quad \text{and} \quad A(u_n)\to A(u) \quad \text{in } W_0^{1,\mathcal{H}}(\Omega)^*,$$

due to the continuity of A, see Proposition 2.1.

Since $\xi_n \in \widetilde{N}_f(u_n)$ we have $\xi_n(x)\in f(x,u_n(x),\nabla u_n(x))$ for a.a. $x\in\Omega$. However, (3.5) and (3.6) imply that the sequence $\{\xi_n\}$ is bounded in $L^{q_1'}(\Omega)$. Passing to a subsequence if necessary, we may suppose that $\xi_n \rightharpoonup \xi$ in $L^{q_1'}(\Omega)$ for some $\xi\in L^{q_1'}(\Omega)$. Employing Mazur's theorem, we are able to find a sequence $\{\eta_n\}$ of convex combinations of $\{\xi_n\}$ such that

$$\eta_n \to \xi \quad \text{in } L^{q_1'}(\Omega).$$

Therefore, we can say that

$$\eta_n(x)\to\xi(x) \quad \text{for a.a. } x\in\Omega, \tag{3.8}$$

see Migórski-Ochal-Sofonea [23, Theorem 2.39].

From (3.6) and condition H(f)(iii) we see that the sequence $\{\xi_n(x)\}$ is bounded for a.a. $x\in\Omega$. So, by (3.8), we find a subsequence $\{\xi_n(x)\}$ for a.a. $x\in\Omega$, still denoted by $\{\xi_n(x)\}$, such that

$$\xi_n(x)\to\xi(x) \quad \text{as } n\to\infty.$$

Keeping in mind that $u_n\to u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)\subset W_0^{1,p}(\Omega)$ leads to

$$u_n(x)\to u(x) \quad \text{and} \quad \nabla u_n(x)\to\nabla u(x) \quad \text{as } n\to\infty.$$

Combining the convergence properties above along with the upper semicontinuity of $(s,\zeta)\mapsto f(x,s,\zeta)$ and Proposition 3.12 in Migórski-Ochal-Sofonea [23] we obtain

$$\xi(x)\in f(x,u(x),\nabla u(x)) \quad \text{for a.a. } x\in\Omega.$$

This means that $\xi\in\widetilde{N}_f(u)$, namely, $i^*\xi\in N_f(u)$. Therefore, we have $u^*=A(u)+i^*\xi\in\mathcal{A}(u)$ which implies that $\mathcal{A}$ is generalized pseudomonotone.

Because $\mathcal{A}$ is a bounded operator with nonempty, closed and convex values, we are now in the position to apply Proposition 2.3 in order to conclude that $\mathcal{A}$ is a pseudomonotone operator.

Furthermore, we are going to prove that there exists a constant $R>0$ such that

$$\langle u^*+\eta,u\rangle_{\mathcal{H}}>0 \tag{3.9}$$

for all $u^*\in\mathcal{A}(u)$, for all $\eta\in\partial I_K(u)$ and for all $u\in W_0^{1,\mathcal{H}}(\Omega)$ with $\|u\|_{1,\mathcal{H},0}=R$.

For any $u^* \in \mathcal{A}(u)$, we can find $\xi \in \widetilde{N}_f(u)$ such that $u^* = A(u) - i^*\xi$. Recall that $0 \in K$, one has

$$\begin{aligned}\langle u^* + \eta, u\rangle_{\mathcal{H}} &\geq \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u\,dx + \int_\Omega \mu(x)|\nabla u|^{q-2}\nabla u\cdot\nabla u\,dx \\ &\quad - \int_\Omega \xi(x)u(x)\,dx + I_K(u) - I_K(0) \\ &\geq \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u\,dx + \int_\Omega \mu(x)|\nabla u|^{q-2}\nabla u\cdot\nabla u\,dx \\ &\quad - \int_\Omega \xi(x)u(x)\,dx + I_K(u) \\ &\geq \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q - \int_\Omega \xi(x)u(x)\,dx + I_K(u).\end{aligned} \tag{3.10}$$

Note that $I_K\colon W_0^{1,\mathcal{H}} \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function. Hence, we can apply Proposition 1.3.1 in Gasiński-Papageorgiou [13] to find $a_K, b_K > 0$ such that

$$I_K(v) \geq -a_K\|v\|_{1,\mathcal{H},0} - b_K \quad \text{for all } v \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.11}$$

Additionally, hypothesis H(f)(iv) implies

$$\int_\Omega \xi(x)u(x)\,dx \leq b_1\|\nabla u\|_p^p + b_2\|u\|_p^p + \|w\|_1. \tag{3.12}$$

Applying (3.11) and (3.12) in (3.10) and taking $W_0^{1,\mathcal{H}}(\Omega) \subseteq W_0^{1,p}(\Omega)$ as well as

$$\|u\|_p^p \leq \lambda_{1,p}^{-1}\|\nabla u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

into account, we get

$$\begin{aligned}&\langle u^* + \eta, u\rangle_{\mathcal{H}} \\ &\geq \|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q - b_1\|\nabla u\|_p^p - b_2\|u\|_p^p - \|w\|_1 - a_K\|u\|_{1,\mathcal{H},0} - b_K \\ &\geq \left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right)\|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q - \|w\|_1 - a_K\|u\|_{1,\mathcal{H},0} - b_K \\ &\geq \left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right)\left(\|\nabla u\|_p^p + \|\nabla u\|_{q,\mu}^q\right) - \|w\|_1 - a_K\|u\|_{1,\mathcal{H},0} - b_K \\ &\geq \left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right)\min\left\{\|u\|_{1,\mathcal{H},0}^p, \|u\|_{1,\mathcal{H},0}^q\right\} - \|w\|_1 - a_K\|u\|_{1,\mathcal{H},0} - b_K,\end{aligned}$$

where the last inequality is obtained by using inequality (2.2). Since $1 < p < q < N$ and $b_1 + b_2\lambda_{1,p}^{-1} < 1$, we can take $R_0 > 0$ large enough such that for all $R \geq R_0$ it holds

$$\left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right)\min\{R^p, R^q\} - \|w\|_1 - a_K R - b_K > 0.$$

Therefore, inequality (3.9) is valid.

Note that $\partial I_K\colon W_0^{1,\mathcal{H}}(\Omega) \to 2^{W_0^{1,\mathcal{H}}(\Omega)^*}$ is a maximal monotone operator. Therefore, we can apply Theorem 2.4 for $F = \partial I_K$, $G = \mathcal{A}$ and $L = 0$. This shows that inclusion (3.4) has at least one solution $u \in K$ which is a solution of (3.3) and so, a solution from (1.1) in the sense of Definition 3.2. Thus, $\mathcal{S} \neq \emptyset$.

Next, we are going to show that the set of solutions of problem (1.1) is closed in $W_0^{1,\mathcal{H}}(\Omega)$. Let $\{u_n\} \subset \mathcal{S}$ be a sequence such that

$$u_n \to u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \tag{3.13}$$

for some $u \in W_0^{1,\mathcal{H}}(\Omega)$. So, for each $n \in \mathbb{N}$, there exists $\xi_n \in \widetilde{N}_f(u_n)$ such that

$$\langle A(u_n), v - u_n\rangle_{\mathcal{H}} + \langle \xi_n, v - u_n\rangle_{L^{q_1}(\Omega)} + I_K(v) - I_K(u_n) \geq 0 \tag{3.14}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Hypothesis H($f$)(iii) and the convergence in (3.13) ensure that $\{\xi_n\}$ is bounded in $L^{q_1'}(\Omega)$. So, we may assume that

$$\xi_n \rightharpoonup \xi \quad \text{in } L^{q_1'}(\Omega).$$

As before, from Mazur's theorem and the upper semicontinuity of $(s,\eta) \mapsto f(x,s,\eta)$, we can show that

$$\xi(x) \in f(x, u(x), \nabla u(x)) \quad \text{for a.a. } x \in \Omega,$$

that is, $\xi \in \widetilde{N}_f(u)$. Passing to the upper limit in (3.14) as $n \to \infty$ and taking the lower semicontinuity of I_K into account it follows that $u \in K$ is a solution of problem (1.1). Hence, $\mathcal{S}$ is closed.

In the last part of the proof we need to show that $\mathcal{S}$ is bounded. If K is bounded, the desired conclusion holds automatically. Let us suppose that K is unbounded and in addition, let us assume that $\mathcal{S}$ is unbounded. Then, there exists a sequence $\{u_n\} \subseteq \mathcal{S}$ such that

$$\|u_n\|_{1,\mathcal{H},0} \to +\infty. \tag{3.15}$$

As before, see (3.10), we can show via a simple calculation that

$$\begin{aligned} 0 &\geq \langle A(u_n) - i^*\xi_n, u_n\rangle_{\mathcal{H}} \\ &\geq \left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right) \min\left\{\|u_n\|_{1,\mathcal{H},0}^p, \|u_n\|_{1,\mathcal{H},0}^q\right\} - \|w\|_1 - a_K\|u_n\|_{1,\mathcal{H},0} - b_K \end{aligned}$$

for some $\xi_n \in \widetilde{N}_f(u_n)$ where we have used the fact that $0 \in K$. Combining the inequality above and (3.15) yields a contradiction. Therefore, $\mathcal{S}$ is bounded. □

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