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Singular p-Laplacian equations with superlinear perturbation

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Abstract

We consider a nonlinear Dirichlet problem driven by the p-Laplace operator and with a right-hand side which has a singular term and a parametric superlinear perturbation. We are interested in positive solutions and prove a bifurcation-type theorem describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. In addition, we show that for every admissible parameter $\lambda > 0$ the problem has a smallest positive solution $\overline{u}_\lambda$ and we establish the monotonicity and continuity properties of the map $\lambda \to \overline{u}_\lambda$.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we deal with the following nonlinear parametric singular problem

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$$
\begin{aligned}
-\Delta_p u &= u^{-\gamma} + \lambda f(x,u) && \text{in } \Omega,\\
u &> 0 && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{P$_\lambda$}
$$

where $1 < p < \infty$, $0 < \gamma < 1$ and Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

In the right-hand side of (P$_\lambda$), $u^{-\gamma}$ is the singular term while λf is the parametric term with $\lambda > 0$ and a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, that is, $x \to f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \to f(x,s)$ is continuous for a.a. $x \in \Omega$. We assume that $f(x,\cdot)$ exhibits $(p-1)$-superlinear growth near $+\infty$ but without satisfying the usual Ambrosetti–Rabinowitz condition, AR-condition for short. We are interested in finding positive solutions and our goal is to determine how the set of positive solutions of (P$_\lambda$) changes as the parameter $\lambda > 0$ varies. We are going to prove a bifurcation-type result which produces a critical parameter value $\lambda^* > 0$ such that

- problem (P$_\lambda$) has at least two positive solutions for all $\lambda \in (0, \lambda^*)$;
- problem (P$_\lambda$) has at least one positive solution for $\lambda = \lambda^*$;
- problem (P$_\lambda$) has no positive solutions for all $\lambda > \lambda^*$.

This result was motivated by the work of Papageorgiou–Smyrlis [15] who proved such a theorem for problem (P$_\lambda$) under the hypotheses that the perturbation term $f(x,\cdot)$ is $(p-1)$-linear near 0^+. This condition removes from consideration nonlinearities with a concave term near 0^+. Our framework removes this restriction and incorporates perturbations which exhibit the competing effects of concave and convex terms. This changes the geometry of the problem. Moreover, our growth condition on $f(x,\cdot)$ is more general than that in Papageorgiou–Smyrlis [15].

Nonlinear singular Dirichlet problems were also investigated in the papers of Giacomoni–Schindler–Takáč [5], Papageorgiou–Rădulescu–Repovš [14] and Perera–Zhang [16] for different settings and conditions.

2. Preliminaries and hypotheses

Let X be a Banach space and let X^* be its topological dual. We denote by $\langle\cdot,\cdot\rangle$ the duality brackets to the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$ we say that φ satisfies the Cerami condition, C-condition for short, if every sequence $\{u_n\}_{n\ge1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\ge1} \subseteq \mathbb{R}$ is bounded and such that $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ and leads to following minimax theorem known as the mountain pass theorem.

Theorem 2.1. *Let $\varphi \in C^1(X,\mathbb{R})$ be a functional satisfying the C-condition and let $u_1, u_2 \in X$, $\|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: \eta_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$ *with* $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$. *Then* $c \ge \eta_\rho$ *with* c *being a critical value of* φ, *that is, there exists* $\hat{u} \in X$ *such that* $\varphi'(\hat{u}) = 0$ *and* $\varphi(\hat{u}) = c$.

By $W_0^{1,p}(\Omega)$ we denote the usual Sobolev space with norm $\|\cdot\|$. Thanks to the Poincaré inequality we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$ and $L^p\left(\Omega;\mathbb{R}^N\right)$, respectively. Furthermore, we need the ordered Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$ and its positive cone

$$C_0^1(\overline{\Omega})_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) = \left\{u \in C_0^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \Omega \text{ and } \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0\right\},$$

where n is the outward unit normal on $\partial\Omega$.

The norm of $\mathbb{R}^N$ is denoted by $|\cdot|$ and "$\cdot$" stands for the inner product in $\mathbb{R}^N$. For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is well known that

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

For $u, v \in W_0^{1,p}(\Omega)$ with $u(x) \le v(x)$ for a.a. $x \in \Omega$ we define

$$[u,v] = \left\{y \in W_0^{1,p}(\Omega) : u(x) \le y(x) \le v(x) \text{ for a.a. } x \in \Omega\right\},$$

$$\operatorname{int}_{C_0^1(\overline{\Omega})} [u,v] = \text{the interior in } C_0^1(\overline{\Omega}) \text{ of } [u,v] \cap C_0^1(\overline{\Omega}),$$

$$[u) = \left\{y \in W_0^{1,p}(\Omega) : u(x) \le y(x) \text{ for a.a. } x \in \Omega\right\}.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. By $p^* > 1$ we denote the Sobolev critical exponent for p defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \le p. \end{cases}$$

Finally, if $h_1, h_2 \in L^\infty(\Omega)$, then we write $h_1 \prec h_2$ if and only if for every compact $K \subseteq \Omega$ we have $0 < m_K \le h_2(x) - h_1(x)$ for a.a. $x \in K$.

Let $A : W_0^{1,p}(\Omega) \to W^{-1.p'}(\Omega) = W_0^{1,p}(\Omega)^*$ with $\frac{1}{p} + \frac{1}{p'} = 1$ be defined by

$$\langle A(u), \varphi\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx \quad \text{for all } u, \varphi \in W_0^{1,p}(\Omega). \tag{2.1}$$

The next proposition states the main properties of this map and it can be found in Gasiński–Papageorgiou [4, Problem 2.192, p. 279].

Proposition 2.2. *The map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined in (2.1) is bounded, that is, it maps bounded sets to bounded sets, continuous, strictly monotone, hence maximal monotone and it is of type* $(\mathrm{S})_+$*, that is,*

$$u_n \xrightarrow{\mathrm{w}} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0,$$

imply $u_n \to u$ in $W_0^{1,p}(\Omega)$.

Moreover, we denote by $\hat{\lambda}_1$ the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and by $\hat{u}_1 \in W_0^{1,p}(\Omega)$ the corresponding positive, L^p-normalized, that is, $\|\hat{u}_1\|_p = 1$, eigenfunction. We know that $\hat{\lambda}_1 > 0$ and $\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Gasiński–Papageorgiou [3].

Also, for a given $\varphi \in C^1(X, \mathbb{R})$ we denote by K_φ the critical set of φ, that is, $K_\varphi = \{u \in X : \varphi'(u) = 0\}$.

Now we introduce the hypotheses on the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x,0) = 0$ for a.a. $x \in \Omega$ and

(i) if $a \in L^s(\Omega)$ with $s > N$, then

$$0 < f(x,s) \le a(x)\left(1 + s^{r-1}\right)$$

for a.a. $x \in \Omega$, for all $s > 0$ and for $p < r < p^*$;

(ii) if $F(x,s) = \int_0^s f(x,t)dt$, then

$$\lim_{s\to+\infty} \frac{F(x,s)}{s^p} = +\infty \quad \text{uniformly for a.a. } x \in \Omega;$$

(iii) if

$$\hat{\eta}_\lambda(x,s) = \left[1 - \frac{p}{1-\gamma}\right] s^{1-\gamma} + \lambda\left[f(x,s)s - pF(x,s)\right]$$

with $\lambda > 0$, then

$$\hat{\eta}_\lambda(x,s_1) \le \hat{\eta}_\lambda(x,s_2) + \tau_\lambda(x)$$

for a.a. $x \in \Omega$, for all $0 \le s_1 \le s_2$ with $\tau_\lambda \in L^1(\Omega)$ and $\lambda \to \tau_\lambda$ is nondecreasing from $(0,+\infty)$ into $L^1(\Omega)$;

(iv) there exist $c_1 > 0$ and $q \le p$ such that

$$f(x,s) \le c_1\left[s^{r-1} + s^{q-1}\right]$$

for a.a. $x \in \Omega$ and for all $s \ge 0$;

(v) for every $\eta > 0$ there exists $m_\eta > 0$ such that

$$f(x,s) \geq m_\eta$$

for a.a. $x \in \Omega$ and for all $s \geq \eta$;

(vi) for every $\rho > 0$ there exists $\hat{\xi}_\rho > 0$ such that the function

$$s \to f(x,s) + \hat{\xi}_\rho s^{p-1}$$

is nondecreasing on $[0, \rho]$ for a.a. $x \in \Omega$.

Remark 2.3. Since we are interested on positive solutions and the hypotheses above concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

$$f(x,s) = 0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \leq 0. \tag{2.2}$$

Hypotheses H(ii), H(iii) imply that

$$\lim_{s \to +\infty} \frac{f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$

Hence, the perturbation term in (P_λ) is $(p-1)$-superlinear in the second variable. However, we do not employ the usual AR-condition for superlinear problems. Recall that this condition says that there exist $\tau > p$ and $M > 0$ such that

$$0 < \tau F(x,s) \leq f(x,s)s \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M, \tag{2.3}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M). \tag{2.4}$$

In fact this is a unilateral version of the AR-condition on account of (2.2). Integrating (2.3) and using (2.4) we obtain the weaker condition

$$c_2 s^\tau \leq F(x,s) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \geq M \text{ and for some } c_2 > 0.$$

Hence, the AR-condition implies that $f(x,\cdot)$ exhibits at least $(\tau - 1)$-polynomial growth. This excludes superlinear nonlinearities with slower growth near $+\infty$ from consideration. Instead we employ the quasimonotonicity condition on $\eta_\lambda(x,\cdot)$ in hypothesis H(iii). This condition is a slight generalization of a hypothesis introduced by Li–Yang [11]. This superlinearity hypothesis is different from the one used by Papageorgiou–Smyrlis [15]. There are easy ways to verify H(iii). For example, condition H(iii) holds if there exists $M > 0$ such that

$$s \to \frac{s^{-\gamma} + \lambda f(x,s)}{s^{p-1}}$$

is nondecreasing on $[M, +\infty)$ for a.a. $x \in \Omega$ or

$$s \to \hat{\eta}_\lambda(x,s)$$

is nondecreasing on $[M, +\infty)$, see Li–Yang [11].

Hypothesis H(iv) allows perturbations which have concave terms. This is excluded from the hypotheses of Papageorgiou–Smyrlis [15]. Hypothesis H(iv) is satisfied if, for example, $f(x,\cdot)$ is differentiable for a.a. $x \in \Omega$ and for every $\rho > 0$ there exists $c_\rho > 0$ such that

$$f_s'(x,s) \geq -c_\rho s^{p-1}$$

for a.a. $x \in \Omega$ and for all $0 \leq s \leq \rho$.

Example 2.4. For the sake of simplicity we drop the x-dependence. The following functions satisfy hypotheses H:

$$f_1(s) = s^{\tau-1} \text{ with } p < \tau < p^*,$$
$$f_2(s) = \begin{cases} (s^+)^{\vartheta-1} & \text{if } s \leq 1, \\ s^{p-1}[\ln s + 1] & \text{if } 1 < s \end{cases} \quad \text{with } 1 < \vartheta < p < \infty.$$

Note that f_2 fails to satisfy the AR-condition and it is outside the framework of Papageorgiou–Smyrlis [15].

3. Positive solutions

We introduce the following two sets

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (\mathrm{P}_\lambda) \text{ has a positive solution}\},$$
$$\mathcal{S}_\lambda = \{u : u \text{ is a positive solution of problem } (\mathrm{P}_\lambda)\}.$$

Proposition 3.1. *If hypotheses H hold, then* $\mathcal{L} \neq \emptyset$.

Proof. We consider the following purely singular Dirichlet problem

$$-\Delta_p u = u^{-\gamma} \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0, \quad u > 0. \tag{3.1}$$

From Papageorgiou–Smyrlis [15, Proposition 5] we know that problem (3.1) has a unique positive solution $\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Moreover, we consider the following auxiliary Dirichlet problem

$$-\Delta_p u = 1 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{3.2}$$

Problem (3.2) has a unique solution $e \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ which can be shown easily. For $1 < \tau < +\infty$, we have $e^\tau \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and using Proposition 2.1 of Marano–Papageorgiou [12], see also Gasiński–Papageorgiou [4, Problem 4.180, p. 680], there exists $c_3 > 0$ such that $\hat{u}_1 \leq c_3 e^\tau$ and so

$$\hat{u}_1^{\frac{1}{\tau}} \leq c_3^{\frac{1}{\tau}} e,$$

which implies

$$e^{-\gamma} \le c_4 \hat{u}_1^{-\frac{\gamma}{\tau}} \tag{3.3}$$

for some $c_4 > 0$. From the Lemma in Lazer–McKenna [9] we know that

$$\hat{u}_1^{-\frac{\gamma}{\tau}} \in L^\tau(\Omega).$$

This fact along with (3.3) gives

$$e^{-\gamma} \in L^\tau(\Omega) \quad \text{and} \quad \left\| e^{-\gamma} \right\|_\tau \le c_4 \left\| \hat{u}_1^{-\gamma} \right\|_1^{\frac{1}{\tau}}.$$

Hence

$$\limsup_{\tau \to +\infty} \left\| e^{-\gamma} \right\|_\tau \le c_4. \tag{3.4}$$

On the other hand, from the Chebyshev inequality, we have

$$\eta^\tau \left| \left\{ e^{-\gamma} \ge \eta \right\} \right|_N \le \left\| e^{-\gamma} \right\|_\tau^\tau$$

with $\eta > 0$, or equivalently,

$$\eta \left| \left\{ e^{-\gamma} \ge \eta \right\} \right|_N^{\frac{1}{\tau}} \le \left\| e^{-\gamma} \right\|_\tau.$$

This fact yields

$$\eta \le \liminf_{\tau \to +\infty} \left\| e^{-\gamma} \right\|_\tau \quad \text{provided} \quad \left| \left\{ e^{-\gamma} \ge \eta \right\} \right|_N > 0. \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$e^{-\gamma} \in L^\infty(\Omega) \quad \text{and} \quad \left\| e^{-\gamma} \right\|_\tau \to \left\| e^{-\gamma} \right\|_\infty \quad \text{as } \tau \to +\infty.$$

Now let $c_5 > \left\| e^{-\gamma} \right\|_\infty$ and $m_0 = \|e\|_\infty$. For $t > 0$ we consider the function

$$\begin{aligned}
\vartheta(t) &= \frac{t^{p-1} - c_5 t^{-\gamma}}{c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1} \right]} \\
&= \frac{t^{p+\gamma-1} - c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1} \right]} \\
&= \frac{1}{c_1 \left[m_0^{r-1} t^{r-p} + m_0^{q-1} t^{q-p} \right]} - \frac{c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1} \right]} \\
&= \frac{t^{p-q}}{c_1 \left[m_0^{r-1} t^{r-q} + m_0^{q-1} \right]} - \frac{c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1} \right]}.
\end{aligned}$$

Since $q \leq p < r$ we see that

$$\vartheta(t) \to -\infty \text{ as } t \to 0^+ \quad \text{and} \quad \vartheta(t) \to 0^+ \text{ as } t \to +\infty.$$

Therefore, there exists $t_0 > 0$ such that

$$\lambda_0 = \vartheta(t_0) = \max[\vartheta(t) : t > 0] > 0.$$

Let $\lambda \in (0, \lambda_0)$. We can find $t > 0$ such that $\vartheta(t) \geq \lambda$. Hence

$$t^{p-1} \geq c_5 t^{-\gamma} + \lambda c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1}\right]. \tag{3.6}$$

We set $\overline{u} = te \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Then, because of (3.6), hypothesis H(iv) and the choice of c_5, m_0, we obtain

$$\begin{aligned} -\Delta_p \overline{u} &= t^{p-1}\left[-\Delta_p e\right] \\ &= t^{p-1} \\ &\geq c_5 t^{-\gamma} + \lambda c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1}\right] \\ &\geq \overline{u}^{-\gamma} + \lambda c_1 \left[\overline{u}^{r-1} + \overline{u}^{q-1}\right] \\ &\geq \overline{u}^{-\gamma} + \lambda f(x, \overline{u}) \quad \text{for a.a. } x \in \Omega. \end{aligned} \tag{3.7}$$

Since $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, as before, there exists $\vartheta \in (0,1)$ small enough such that $\vartheta \tilde{u} \leq \overline{u}$. If $\tilde{u}_0 = \vartheta \tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, then

$$-\Delta_p \tilde{u}_0 = -\Delta_p(\vartheta \tilde{u}) = \vartheta^{p-1}\left(-\Delta_p \tilde{u}\right) = \vartheta^{p-1} \tilde{u}^{-\gamma} \leq (\vartheta \tilde{u})^{-\gamma} = \tilde{u}_0^{-\gamma} \tag{3.8}$$

since $\vartheta \in (0,1)$. Using the functions $\tilde{u}_0, \overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, we introduce the following truncation of the reaction of problem (P_λ)

$$g_\lambda(x,s) = \begin{cases} \tilde{u}_0(x)^{-\gamma} + \lambda f(x, \tilde{u}_0(x)) & \text{if } s < \tilde{u}_0(x), \\ s^{-\gamma} + \lambda f(x,s) & \text{if } \tilde{u}_0(x) \leq s \leq \overline{u}(x), \\ \overline{u}(x)^{-\gamma} + \lambda f(x, \overline{u}(x)) & \text{if } \overline{u}(x) < s, \end{cases} \tag{3.9}$$

with $\lambda \in (0, \lambda_0)$. Evidently, $g_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We set $G_\lambda(x,s) = \int_0^s g_\lambda(x,t)dt$ and consider the functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega G_\lambda(x,u)dx.$$

On account of Proposition 3 of Papageorgiou–Smyrlis [15] we have that $\psi_\lambda \in C^1(W_0^{1,p}(\Omega))$. Moreover, from (3.9) it is clear that ψ_λ is coercive. The Sobolev embedding theorem implies that

ψ_λ is sequentially weakly lower semicontinuous. So, by the Weierstraß–Tonelli theorem, there exists $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_\lambda) = \inf\left[\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)\right].$$

Since u_λ is a global minimizer, it fulfills $\psi_\lambda'(u_\lambda) = 0$, which is equivalent to

$$\langle A(u_\lambda), h\rangle = \int_\Omega g_\lambda(x, u_\lambda) h dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.10}$$

Taking $h = (\tilde{u}_0 - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.10) gives, thanks to (3.9), (3.8) and the fact that $f \geq 0$,

$$\begin{aligned}\left\langle A(u_\lambda), (\tilde{u}_0 - u_\lambda)^+\right\rangle &= \int_\Omega \left[\tilde{u}_0^{-\gamma} + \lambda f(x, \tilde{u}_0)\right](\tilde{u}_0 - u_\lambda)^+ dx \\ &\geq \int_\Omega \tilde{u}_0^{-\gamma}(\tilde{u}_0 - u_\lambda)^+ dx \\ &\geq \left\langle A(\tilde{u}_0), (\tilde{u}_0 - u_\lambda)^+\right\rangle.\end{aligned}$$

Because of the monotonicity of A, see Proposition 2.2, we obtain that $\tilde{u}_0 \leq u_\lambda$. Next, we choose $h = (u_\lambda - \overline{u})^+ \in W_0^{1,p}(\Omega)$ in (3.10). This gives, by applying (3.9) and (3.7), that

$$\left\langle A(u_\lambda), (u_\lambda - \overline{u})^+\right\rangle = \int_\Omega \left[\overline{u}^{-\gamma} + \lambda f(x, \overline{u})\right](u_\lambda - \overline{u})^+ dx \leq \left\langle A(\overline{u}), (u_\lambda - \overline{u})^+\right\rangle.$$

As before, by applying Proposition 2.2, it follows that $u_\lambda \leq \overline{u}$. So, we have proved that

$$u_\lambda \in \left[\tilde{u}_0, \overline{u}\right]. \tag{3.11}$$

From (3.9), (3.10), (3.11), it follows that

$$\langle A(u_\lambda), h\rangle = \int_\Omega \left[u_\lambda^{-\gamma} + \lambda f(x, u_\lambda)\right] h dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.12}$$

Since $\tilde{u}_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, as before, we have that $\tilde{u}_0^{-\gamma} \in L^s(\Omega)$ for $s > N$ and since $0 \leq u_\lambda^{-\gamma} \leq \tilde{u}_0^{-\gamma}$, see (3.11), one has that $u_\lambda^{-\gamma} \in L^s(\Omega)$. From (3.12) it follows that

$$-\Delta_p u_\lambda(x) = u_\lambda(x)^{-\gamma} + \lambda f(x, u_\lambda(x)) \quad \text{for a.a. } x \in \Omega, \quad u_\lambda\big|_{\partial\Omega} = 0. \tag{3.13}$$

From (3.13) and Proposition 1.3 of Guedda–Véron [7] we have that $u_\lambda \in L^\infty(\Omega)$. Let $\xi_\lambda(x) = u_\lambda(x)^{-\gamma} + \lambda f(x, u_\lambda(x))$. Then $\xi_\lambda \in L^s(\Omega)$, see hypothesis H(i). We consider now the following linear Dirichlet problem

$$-\Delta v = \xi_\lambda \quad \text{in } \Omega, \quad v\big|_{\partial\Omega}.$$

This problem has a unique solution v_λ which by the linear regularity theory belongs to $W^{2,s}(\Omega)$, see Gilbarg–Trudinger [6, Theorem 9.15, p. 241]. Then, since $s > N$, the Sobolev embedding theorem implies that

$$v_\lambda \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{with} \quad \alpha = 1 - \frac{N}{s}. \tag{3.14}$$

We set $k_\lambda(x) = \nabla v_\lambda(x)$. Then $k_\lambda \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$, see (3.14). From (3.13) we obtain

$$-\operatorname{div}\left(|\nabla u_\lambda(x)|^{p-2}\nabla u_\lambda(x) - k_\lambda(x)\right) = 0 \quad \text{for a.a.}\, x \in \Omega, \quad u_\lambda\big|_{\partial\Omega} = 0.$$

Invoking Theorem 1 of Lieberman [10], we infer that $u_\lambda \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Finally from (3.13) and the nonlinear maximum principle, see for example, Gasiński–Papageorgiou [3, Theorem 6.2.8, p. 738] and Pucci–Serrin [17, p. 120], we conclude that $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. It follows that $(0, \lambda_0) \subseteq \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$. □

From the proof above we infer the following corollary.

Corollary 3.2. *If hypotheses H hold and $\lambda \in \mathcal{L}$, then $\mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

In the next proposition we show that $\mathcal{L}$ is in fact an interval.

Proposition 3.3. *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$ there exists $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Corollary 3.2. Since $\mu < \lambda$ and $f \geq 0$, we have

$$-\Delta_p u_\lambda(x) = u_\lambda(x)^{-\gamma} + \lambda f(x, u_\lambda(x)) \geq u_\lambda(x)^{-\gamma} + \mu f(x, u_\lambda(x))$$

for a.a. $x \in \Omega$. Recall that $\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique solution of (3.1). Since $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ there exists $t \in (0, 1)$ small enough such that $t\tilde{u} \leq u_\lambda$. We set $\tilde{u}_* = t\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and introduce the following truncation nonlinearity

$$\hat{g}_\mu(x, s) = \begin{cases} \tilde{u}_*(x)^{-\gamma} + \mu f\,(x, \tilde{u}_*(x)) & \text{if } s < \tilde{u}_*(x), \\ s^{-\gamma} + \mu f(x, s) & \text{if } \tilde{u}_*(x) \leq s \leq u_\lambda(x), \\ u_\lambda(x)^{-\gamma} + \mu f(x, u_\lambda(x)) & \text{if } u_\lambda(x) < s, \end{cases} \tag{3.15}$$

which is a Carathéodory function. We set $\hat{G}_\mu(x, s) = \int_0^s \hat{g}_\mu(x, t)dt$ and consider the functional $\hat{\psi}_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\mu(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \hat{G}_\mu(x, u)dx.$$

As before, we have $\hat{\psi}_\mu \in C^1(W_0^{1,p}(\Omega))$, see Papageorgiou–Smyrlis [15, Proposition 3]. From (3.15) it is clear that $\hat{\psi}_\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstraß–Tonelli theorem there exists $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_\mu\left(u_\mu\right) = \inf\left[\hat{\psi}_\mu(u) : u \in W_0^{1,p}(\Omega)\right].$$

Hence, $\hat{\psi}'_\mu(u_\mu) = 0$ which is equivalent to

$$\langle A(u_\mu), h\rangle = \int_\Omega \hat{g}_\mu(x, u_\mu) h dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.16}$$

We choose $h = \left(\tilde{u}_* - u_\mu\right)^+ \in W_0^{1,p}(\Omega)$ in (3.16). Then, using (3.15), $f \geq 0$, (3.1) and $\tilde{u}_* = t\tilde{u}$ for $0 < t < 1$, we obtain

$$\begin{aligned}\langle A(u_\mu), \left(\tilde{u}_* - u_\mu\right)^+\rangle &= \int_\Omega \left[\tilde{u}_*^{-\gamma} + \mu f\left(x, \tilde{u}_*\right)\right]\left(\tilde{u}_* - u_\mu\right)^+ dx \\ &\geq \int_\Omega \tilde{u}_*^{-\gamma}\left(\tilde{u}_* - u_\mu\right)^+ dx \\ &\geq \left\langle A\left(\tilde{u}_*\right), \left(\tilde{u}_* - u_\mu\right)^+\right\rangle.\end{aligned}$$

Hence, by Proposition 2.2, $\tilde{u}_* \leq u_\mu$. Next, we choose $h = \left(u_\mu - u_\lambda\right)^+ \in W_0^{1,p}(\Omega)$ in (3.16). Then, as before, by applying (3.15) and since $f \geq 0$, $\mu < \lambda$ and $u_\lambda \in \mathcal{S}_\lambda$ we obtain

$$\begin{aligned}\left\langle A\left(u_\mu\right), \left(u_\mu - u_\lambda\right)^+\right\rangle &= \int_\Omega \left[u_\lambda^{-\gamma} + \mu f\left(x, u_\mu\right)\right]\left(u_\mu - u_\lambda\right)^+ dx \\ &\leq \left[u_\lambda^{-\gamma} + \lambda f\left(x, u_\lambda\right)\right]\left(u_\mu - u_\lambda\right)^+ dx \\ &= \left\langle A\left(u_\lambda\right), \left(u_\mu - u_\lambda\right)^+\right\rangle.\end{aligned}$$

Using Proposition 2.2 we see that $u_\mu \leq u_\lambda$.

So, we have proved that

$$u_\mu \in \left[\tilde{u}_*, u_\lambda\right]. \tag{3.17}$$

From (3.15), (3.16) and (3.17) we infer that $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so $\mu \in \mathcal{L}$. □

A useful byproduct of the proof above is the following corollary.

Corollary 3.4. *If hypotheses H hold,* $0 < \mu < \lambda \in \mathcal{L}$ *and* $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, *then* $\mu \in \mathcal{L}$ *and there exists* $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *such that* $u_\lambda - u_\mu \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$.

In fact using hypotheses H(v), (vi) we can improve the conclusion of the corollary above.

Proposition 3.5. *If hypotheses H hold,* $0 < \mu < \lambda \in \mathcal{L}$ *and if* $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$*, then* $\mu \in \mathcal{L}$ *and there exists* $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *such that* $u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Proof. From Corollary 3.4 we already know that $\mu \in \mathcal{L}$ and we can find $u_\lambda \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_\lambda - u_\mu \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(vi). Since $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, $u_\mu \leq u_\lambda$ and because of hypotheses H(v), (vi) we derive

$$
\begin{aligned}
& -\Delta_p u_\mu(x) + \lambda\hat{\xi}_\rho u_\mu(x)^{p-1} - u_\mu(x)^{-\gamma} \\
& = \mu f\left(x, u_\mu(x)\right) + \lambda\hat{\xi}_\rho u_\mu(x)^{p-1} \\
& = \lambda f\left(x, u_\mu(x)\right) + \lambda\hat{\xi}_\rho u_\mu(x)^{p-1} - (\lambda-\mu) f(x, u_\mu(x)) \\
& < \lambda f\left(x, u_\lambda(x)\right) + \lambda\hat{\xi}_\rho u_\lambda(x)^{p-1} \\
& = -\Delta_p u_\lambda(x) + \lambda\hat{\xi}_\rho u_\lambda^{p-1} - u_\lambda(x)^{-\gamma}
\end{aligned}
\tag{3.18}
$$

for a.a. $x \in \Omega$. Let $\hat{h}_0(x) = (\lambda-\mu) f(x, u_\mu(x))$. Since $u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and using hypothesis H(v), we see that $0 \prec \hat{h}_0$. Therefore, from (3.18) and the singular strong comparison principle, see Papageorgiou–Smyrlis [15, Proposition 4], we conclude that $u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

We set $\lambda^* = \sup \mathcal{L}$.

Proposition 3.6. *If hypotheses H hold, then* $\lambda^* < \infty$.

Proof. Recall that

$$
\lim_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,
$$

see hypotheses H(ii), (iii). Therefore, for a given $k > \hat{\lambda}_1$, there exists $M > 0$ such that

$$
f(x,s) \geq k s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M. \tag{3.19}
$$

On the other hand, we have

$$
s^{-\gamma} + \lambda f(x,s) \geq M^{-\gamma} + \lambda f(x,s) \tag{3.20}
$$

for a.a. $x \in \Omega$, for all $0 \leq s \leq M$ and for all $\lambda > 0$. Note that, since $f \geq 0$,

$$
\lim_{s\to 0^+} \frac{M^{-\gamma} + \lambda f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,
$$

which implies that there exists $\delta_\lambda > 0$ such that

$$M^{-\gamma} + \lambda f(x,s) \ge \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \le s \le \delta_\lambda.$$

Combining this with (3.20) we see that

$$s^{-\gamma} + \lambda f(x,s) \ge \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \le s \le \delta_\lambda. \tag{3.21}$$

Finally, note that on account of hypothesis H(v), there exists $\tilde{\lambda} \ge 1$ large enough such that

$$s^{-\gamma} + \tilde{\lambda} f(x,s) \ge M^{-\gamma} + \tilde{\lambda} m_{\delta_{\tilde{\lambda}}} \ge \hat{\lambda}_1 M^{p-1} \ge \hat{\lambda}_1 s^{p-1} \tag{3.22}$$

for a.a. $x \in \Omega$ and for all $\delta_{\tilde{\lambda}} \le s \le M$. Combining (3.19), (3.21), and (3.22) we conclude that

$$s^{-\gamma} + \tilde{\lambda} f(x,s) \ge \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \ge 0. \tag{3.23}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. There exists $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Let $t > 0$ be such that

$$t\hat{u}_1 \le u_\lambda. \tag{3.24}$$

Assume that $t > 0$ is the largest positive real number for which (3.24) holds. Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(vi). Applying (3.24), hypothesis H(vi) and (3.23) gives

$$\begin{aligned}
&-\Delta_p u_\lambda(x) + \lambda\hat{\xi}_\rho u_\lambda(x)^{p-1} - u_\lambda(x)^{-\gamma}\\
&\quad = \lambda f(x, u_\lambda(x)) + \lambda\hat{\xi}_\rho u_\lambda(x)^{p-1}\\
&\quad \ge \lambda f\left(x, t\hat{u}_1(x)\right) + \lambda\hat{\xi}_\rho \left(t\hat{u}_1(x)\right)^{p-1}\\
&\quad = \tilde{\lambda} f\left(x, t\hat{u}_1(x)\right) + \lambda\hat{\xi}_\rho \left(t\hat{u}_1(x)\right)^{p-1} + \left(\lambda - \tilde{\lambda}\right) f\left(x, t\hat{u}_1(x)\right)\\
&\quad \ge \hat{\lambda}_1 \left(t\hat{u}_1(x)\right)^{p-1} + \lambda\hat{\xi}_\rho \left(t\hat{u}_1(x)\right)^{p-1}\\
&\quad \ge -\Delta_p \left(t\hat{u}_1(x)\right) + \lambda\hat{\xi}_\rho \left(t\hat{u}_1(x)\right)^{p-1} - \left(t\hat{u}_1(x)\right)^{-\gamma} \quad \text{for a.a. } x \in \Omega.
\end{aligned} \tag{3.25}$$

We set $\tilde{h}_0(x) = \left(\lambda - \tilde{\lambda}\right) f\left(x, t\hat{u}_1(x)\right)$. We see that since $\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and because of hypothesis H(v), we have $0 \prec \tilde{h}_0$. Therefore, from (3.25) and Papageorgiou–Smyrlis [15, Proposition 4] we infer that $u_\lambda - t\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ which contradicts the maximality of $t > 0$, see (3.24). This shows that $\lambda \notin \mathcal{L}$ and so $\lambda^* \le \tilde{\lambda} < +\infty$. □

Next we show that the critical parameter $\lambda^* > 0$ is admissible.

Proposition 3.7. *If hypotheses H hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Consider a sequence $\{\lambda_n\}_{n\ge 1} \subseteq (0, \lambda^*) \subseteq \mathcal{L}$ such that $\lambda_n \to (\lambda^*)^-$ as $n \to \infty$. From the proof of Proposition 3.3 we know that there exists $u_n \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for each $n \in \mathbb{N}$ such that

$$\{u_n\}_{n\geq 1} \text{ is increasing and } \tilde{u}_* = t\tilde{u} \leq u_n \quad \text{for all } n \in \mathbb{N}. \tag{3.26}$$

Let $\hat{\psi}_{\lambda_n} \in C^1(W_0^{1,p}(\Omega))$ be as in the proof of Proposition 3.3 resulting from the truncation of the reaction of (P_λ) with λ replaced by λ_n at $\{\tilde{u}_*(x), u_{n+1}(x)\} = \{t\tilde{u}(x), u_{n+1}(x)\}$, see (3.15). We know that $u_n \in [\tilde{u}_*, u_{n+1}]$ is the minimizer of $\hat{\psi}_{\lambda_n}$. Therefore, because of (3.15) with $u_\lambda = u_{n+1}$ and hypothesis H(v), we have

$$\begin{aligned}
\hat{\psi}_{\lambda_n}(u_n) \leq \hat{\psi}_{\lambda_n}(\tilde{u}_*) &= \frac{1}{p}\|\nabla \tilde{u}_*\|_p^p - \int_\Omega \left[\tilde{u}^{1-\gamma} + \lambda_n f(x, \tilde{u}_*)\tilde{u}_*\right] dx \\
&= \frac{t^p}{p}\|\nabla \tilde{u}\|_p^p - t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma} dx - \lambda_n \int_\Omega f(x, \tilde{u}_*)\tilde{u}_* dx \\
&< \frac{t^p}{p}\|\nabla \tilde{u}\|_p^p - t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma} dx.
\end{aligned} \tag{3.27}$$

We know that

$$\|\nabla \tilde{u}\|_p^p = \int_\Omega \tilde{u}^{1-\gamma} dx,$$

see (3.27). Hence, since $t \in (0, 1)$,

$$t^p \|\nabla \tilde{u}\|_p^p \leq t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma} dx.$$

This finally gives

$$\hat{\psi}_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}, \tag{3.28}$$

see (3.27).

Consider now the Carathéodory function $\tilde{g}_{\lambda_n} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{g}_{\lambda_n}(x, s) = \begin{cases} \tilde{u}_*(x)^{-\gamma} + \lambda_n f(x, \tilde{u}_*(x)) & \text{if } s \leq \tilde{u}_*(x), \\ s^{-\gamma} + \lambda_n f(x, s) & \text{if } \tilde{u}_*(x) < s. \end{cases} \tag{3.29}$$

We set $\tilde{G}_{\lambda_n}(x, s) = \int_0^s \tilde{g}_{\lambda_n}(x, t)dt$ and consider the C^1-functional $\tilde{\varphi}_{\lambda_n} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_{\lambda_n}(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \tilde{G}_{\lambda_n}(x, u)dx.$$

Note that

$$\tilde{\varphi}_{\lambda_n}\big|_{[\tilde{u}_*,u_{n+1}]} = \hat{\psi}_{\lambda_n}\big|_{[\tilde{u}_*,u_{n+1}]}.$$

Then, see (3.28), we have $\tilde{\varphi}_{\lambda_n}(u_n) < 0$ for all $n \in \mathbb{N}$ and so

$$\|\nabla u_n\|_p^p - \int_\Omega p\tilde{G}_{\lambda_n}(x,u_n)dx < 0.$$

Applying (3.29) and the fact that $u_n \in \left[\tilde{u}_*, u_{n+1}\right]$ leads to

$$\begin{aligned} &\|\nabla u_n\|_p^p - \int_\Omega p\left[\tilde{u}_*^{1-\gamma} + \lambda_n f(x,\tilde{u}_*)\right]\tilde{u}_* dx \\ &- \frac{p}{1-\gamma}\int_\Omega \left[u_n^{1-\gamma} - u_*^{1-\gamma}\right] - \lambda_n p\int_\Omega \left[F(x,u_n) - F(x,\tilde{u}_*)\right]dx < 0. \end{aligned} \tag{3.30}$$

Moreover, we know that

$$\langle A(u_n), h\rangle = \int_\Omega \tilde{g}_{\lambda_n}(x,u_n)h dx \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ and for all } n \in \mathbb{N}. \tag{3.31}$$

Choosing $h = u_n \in W_0^{1,p}(\Omega)$ in (3.31) and applying (3.29) and the fact that $u_n \in [\tilde{u}_*, u_{n+1}]$ yields

$$-\|\nabla u_n\|_p^p + \int_\Omega \left[u_n^{1-\gamma} + \lambda_n f(x,u_n)u_n\right]dx = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.32}$$

Adding (3.30) and (3.32) we obtain

$$\int_\Omega \hat{\eta}_{\lambda_n}(x,u_n)dx \le M_1 \quad \text{for some } M_1 > 0 \text{ and for all } n \in \mathbb{N}. \tag{3.33}$$

Suppose that $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. By passing to a subsequence if necessary, we may assume that $\|u_n\| \to +\infty$. We set $y_n = \frac{u_n}{\|u_n\|}$ for $n \in \mathbb{N}$. Then we have $\|y_n\| = 1$ and $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \overset{w}{\to} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^r(\Omega), \quad \text{with } y \ge 0. \tag{3.34}$$

First assume that $y \ne 0$ and set $\Omega^* = \{x \in \Omega : \ y(x) > 0\}$. We have $|\Omega^*|_N > 0$ and $u_n(x) \to +\infty$ for all $x \in \Omega^*$. We have

$$\frac{F(x,u_n(x))}{\|u_n\|^p} = \frac{F(x,u_n(x))}{u_n(x)^p}y_n(x)^p \to +\infty \quad \text{for a.a. } x \in \Omega^*$$

and so, by Fatou's Lemma,

$$\int_{\Omega^*} \frac{F(x,u_n)}{\|u_n\|^p} dx \to +\infty. \tag{3.35}$$

Since $F \geq 0$, we have

$$\int_{\Omega^*} \frac{F(x,u_n)}{\|u_n\|^p} dx \leq \int_{\Omega} \frac{F(x,u_n)}{\|u_n\|^p} dx$$

and so, by (3.35),

$$\int_{\Omega} \frac{F(x,u_n)}{\|u_n\|^p} dx \to +\infty. \tag{3.36}$$

Hypothesis H(iii) implies that

$$0 \leq \hat{\eta}_{\lambda_n}(x,u_n(x)) + \tau_{\lambda^*}(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } n \in \mathbb{N}.$$

Then

$$\frac{p}{1-\gamma} u_n(x)^{1-\gamma} + pF(x,u_n(x)) \leq u_n(x)^{1-\gamma} + \lambda_n f(x,u_n(x))u_n(x) + \tau_{\lambda^*}(x) \tag{3.37}$$

for a.a. $x \in \Omega$ and for all $n \in \mathbb{N}$.

From (3.31) with $h = u_n \in W_0^{1,p}(\Omega)$ we obtain by using (3.29) and (3.26)

$$\|\nabla u_n\|_p^p = \int_{\Omega} \left[u_n^{1-\gamma} + \lambda_n f(x,u_n)u_n \right] dx \quad \text{for all } n \in \mathbb{N}. \tag{3.38}$$

Applying (3.38) in (3.37) gives

$$p\lambda_n \int_{\Omega} F(x,u_n) dx \leq \|\nabla u_n\|_p^p + \|\tau_{\lambda^*}\|_1 .$$

Hence

$$p\lambda_n \int_{\Omega} \frac{F(x,u_n)}{\|u_n\|^p} dx \leq \|\nabla y_n\|_p^p + \frac{\|\tau_{\lambda^*}\|_1}{\|u_n\|^p} \quad \text{for all } n \in \mathbb{N}. \tag{3.39}$$

Comparing (3.36) and (3.39) we have a contradiction.

Next suppose that $y = 0$. For $\mu > 0$ we set $v_n = (p\mu)^{\frac{1}{p}} y_n$ for all $n \in \mathbb{N}$. Then $v_n \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $v_n \to 0$ in $L^r(\Omega)$, see (3.34) and recall that $y = 0$. Then, by (3.29), we get

$$\int_{\Omega} \tilde{G}_{\lambda_n}(x,v_n) dx \to 0 \quad \text{as } n \to \infty. \tag{3.40}$$

Since $\|u_n\| \to +\infty$, there exists a number $n_0 \in \mathbb{N}$ such that

$$(p\mu)^{\frac{1}{p}} \frac{1}{\|u_n\|} \leq 1 \quad \text{for all } n \geq n_0. \tag{3.41}$$

Moreover, let $t_n \in [0, 1]$ be such that

$$\tilde{\varphi}_{\lambda_n}(t_n u_n) = \max_{0 \leq t \leq 1} \tilde{\varphi}_{\lambda_n}(t u_n), \qquad n \in \mathbb{N}.$$

Applying (3.41), the representation $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and (3.40) leads to

$$\begin{aligned} \tilde{\varphi}_{\lambda_n}(t_n u_n) &\geq \tilde{\varphi}_{\lambda_n}(v_n) \quad \text{for all } n \geq n_0 \\ &= \mu \|\nabla y_n\|_p^p - \int_\Omega \tilde{G}_{\lambda_n}(x, v_n)dx \\ &= \mu - \int_\Omega \tilde{G}(x, v_n)dx \geq \frac{\mu}{2} \quad \text{for all } n \geq n_1 \geq n_0. \end{aligned} \tag{3.42}$$

But recall that $\mu > 0$ is arbitrary. So, from (3.42) we infer that

$$\tilde{\varphi}_{\lambda_n}(t_n u_n) \to +\infty \quad \text{as } n \to \infty. \tag{3.43}$$

We have

$$\tilde{\varphi}_{\lambda_n}(0) = 0 \quad \text{and} \quad \tilde{\varphi}_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}.$$

From this and (3.43) it follows that $t_n \in (0, 1)$ for all $n \geq n_2$. Therefore, we obtain

$$\frac{d}{dt}\tilde{\varphi}_{\lambda_n}(t u_n)\Big|_{t=t_0} = 0 \quad \text{for all } n \geq n_2$$

which means

$$\|\nabla(t_n u_n)\|_p^p = \int_\Omega \tilde{g}_{\lambda_n}(x, t_n u_n) u_n dx$$

and so

$$p\tilde{\varphi}_{\lambda_n}(t_n u_n) + p\int_\Omega \tilde{G}_{\lambda_n}(x, t_n u_n)dx = \int_\Omega \tilde{g}_{\lambda_n}(x, t_n u_n)(t_n u_n)dx.$$

Then we use hypothesis H(iii), (3.29) and recall that $t_n \in (0, 1)$ for all $n \geq n_2$ to get

$$p\tilde{\varphi}_{\lambda_n}(t_n u_n) \leq \int_\Omega \hat{\eta}_{\lambda_n}(x, u_n)dx + M_2$$

for some $M_2 > 0$ and for all $n \geq n_2$. Taking (3.43) into account gives

$$\int_\Omega \hat{\eta}_{\lambda_n}(x, u_n)dx \to +\infty \quad \text{as } n \to \infty.$$

But this last convergence contradicts (3.33).

It follows that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n \xrightarrow{\mathrm{w}} u^* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u^* \quad \text{in } L^r(\Omega) \quad \text{with } u^* \geq \tilde{u}_*. \tag{3.44}$$

Choosing $h = u_n - u^* \in W_0^{1,p}(\Omega)$ in (3.31), recalling that $u_n^{-\gamma} \in L^{r'}(\Omega)$ with $\frac{1}{r} + \frac{1}{r'} = 1$, passing to the limit as $n \to \infty$ and applying (3.44) results in

$$\lim_{n\to\infty} \langle A(u_n), u_n - u^* \rangle = 0.$$

Since A has the $(\mathrm{S})_+$-property, see Proposition 2.2, we infer that

$$u_n \to u^* \quad \text{in } W_0^{1,p}(\Omega). \tag{3.45}$$

So, if we pass to the limit in (3.31) and apply (3.45), then we obtain

$$\langle A(u^*), h \rangle = \int_\Omega \tilde{g}_{\lambda^*}(x, u^*)hdx \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } u^* \geq \tilde{u}_*.$$

Therefore, we have

$$\langle A(u^*), h \rangle = \int_\Omega \left[(u^*)^{-\gamma} + \lambda^* f(x, u^*)\right] hdx \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Hence, $u^* \in \mathcal{S}_{\lambda^*} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\lambda^* \in \mathcal{L}$. □

In summary, we have proved that

$$\mathcal{L} = \left(0, \lambda^*\right].$$

Next we show that we have two solutions for all $\lambda \in (0, \lambda^*)$.

Proposition 3.8. *If hypotheses H hold and* $0 < \lambda < \lambda^*$, *then problem* (P_λ) *has two positive solutions* $u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Proof. From Proposition 3.7 we know that $\lambda^* \in \mathcal{L}$. So, there exists $u^* \in \mathcal{S}_{\lambda^*} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Corollary 3.2. According to Proposition 3.5 we can find $u_0 \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that

$$u^* - u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.46}$$

Moreover, let $\vartheta \in (0, \lambda) \subseteq \mathcal{L}$ and $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ be such that

$$u_0 - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \tag{3.47}$$

again by Proposition 3.5. From (3.46) and (3.47) it follows that

$$u_0 \in \operatorname*{int}_{C_0^1(\overline{\Omega})} \left[u_\vartheta, u^*\right]. \tag{3.48}$$

We consider the Carathéodory functions $k_\lambda, \hat{k}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_\lambda(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} + \lambda f(x, u_\vartheta(x)) & \text{if } s \le u_\vartheta(x), \\ s^{-\gamma} + \lambda f(x,s) & \text{if } u_\vartheta(x) < s \end{cases} \tag{3.49}$$

and

$$\hat{k}_\lambda(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} + \lambda f(x, u_\vartheta(x)) & \text{if } s < u_\vartheta(x), \\ s^{-\gamma} + \lambda f(x,s) & \text{if } u_\vartheta(x) \le s \le u^*(x), \\ u^*(x)^{-\gamma} + \lambda f(x, u^*(x)) & \text{if } u^*(x) < s. \end{cases} \tag{3.50}$$

We set $K_\lambda(x,s) = \int_0^s k_\lambda(x,t)dt$, $\hat{K}_\lambda(x,s) = \int_0^s \hat{k}_\lambda(x,t)dt$ and consider the C^1-functionals $\sigma_\lambda, \hat{\sigma}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \sigma_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega K_\lambda(x,u)dx, \\ \hat{\sigma}_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \hat{K}_\lambda(x,u)dx. \end{aligned}$$

From (3.49) and (3.50) it is clear that

$$\sigma_\lambda\big|_{[u_\vartheta, u^*]} = \hat{\sigma}_\lambda\big|_{[u_\vartheta, u^*]}. \tag{3.51}$$

Moreover, as in the proof of Proposition 3.1, using (3.49) and (3.50), we show that

$$K_{\sigma_\lambda} \subseteq [u_\vartheta) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad K_{\hat{\sigma}_\lambda} \subseteq [u_\vartheta, u_\lambda] \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.52}$$

From (3.52) we see that we may assume that $K_{\hat{\sigma}_\lambda} = \{u_0\}$, otherwise we already have a second positive solution for problem (P_λ), see (3.50) and (3.52).

From (3.50) and since $u_\vartheta^{-\gamma} \in L^{p'}(\Omega)$ we infer that $\hat{\sigma}_\lambda$ is coercive and from the Sobolev embedding theorem, we know that $\hat{\sigma}_\lambda$ is sequentially weakly lower semicontinuous. Therefore, we can find $u_0^* \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma}_\lambda\left(u_0^*\right) = \inf\left[\hat{\sigma}_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.53}$$

That means $u_0^* \in K_{\hat{\sigma}_\lambda}$ and so $u_0^* = u_0$. From (3.48), (3.51) and (3.53) it follows that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of σ_λ and from [5] and [13] we know that

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \sigma_\lambda. \tag{3.54}$$

We assume that K_{σ_λ} is finite or otherwise, on account of (3.49) and (3.52), we already have an infinity of positive smooth solutions for problem (P_λ) and so we are done. From (3.54) we infer that there exists $\rho \in (0,1)$ small enough such that

$$\sigma_\lambda(u_0) < \inf[\sigma_\lambda(u) : \|u-u_0\| = \rho] = m_\lambda, \tag{3.55}$$

see Aizicovici–Papageorgiou–Staicu [1, Proof of Proposition 29].

Hypothesis H(ii) implies that if $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, then

$$\sigma_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.56}$$

Claim: σ_λ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\ge 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\sigma_\lambda(u_n)| \le M_3 \quad \text{for some } M_3 > 0 \text{ and for all } n \in \mathbb{N}, \tag{3.57}$$

$$(1+\|u_n\|)\,\sigma_\lambda'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.58}$$

From (3.58) we have

$$\left| \langle A(u_n), h\rangle - \int_\Omega k_\lambda(x,u_n) h dx \right| \le \frac{\varepsilon_n \|h\|}{1+\|u_n\|} \tag{3.59}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. We choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ in (3.59) and use (3.49) to obtain

$$\left\|\nabla u_n^-\right\|_p^p \le c_6 \left\|u_n^-\right\| \quad \text{for some } c_6 > 0 \text{ and for all } n \in \mathbb{N}.$$

Hence

$$\{u_n\}_{n\ge 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.60}$$

Then from (3.57) and (3.60) it follows that

$$\left\|\nabla u_n^+\right\|_p^p - \int_\Omega p\hat{K}_\lambda(x,u_n^+) dx \le M_4 \quad \text{for some } M_4 > 0 \text{ and for all } n \in \mathbb{N}.$$

This implies

$$\begin{aligned}
&\|\nabla u_n^+\|_p^p - \int\limits_{\{u_n^+\le u_\vartheta\}} p\left[u_\vartheta^{-\gamma}+\lambda f(x,u_\vartheta)\right]u_n^+dx\\
&-\frac{p}{1-\gamma}\int\limits_{\{u_\vartheta<u_n^+\}}\left[\left(u_n^+\right)^{1-\gamma}-u_\vartheta^{1-\gamma}\right]dx\\
&-p\lambda\int\limits_{\{u_\vartheta<u_n^+\}}\left[F(x,u_n^+)-F(x,u_\vartheta)\right]\le M_4
\end{aligned}$$

for all $n\in\mathbb{N}$ and so

$$\|\nabla u_n^+\|_p^p-\frac{p}{1-\gamma}\int\limits_\Omega\left(u_n^+\right)^{1-\gamma}dx-p\lambda\int\limits_\Omega F(x,u_n^+)dx\le M_5 \tag{3.61}$$

for some $M_5>0$ and for all $n\in\mathbb{N}$. Moreover, we choose $h=u_n^+\in W_0^{1,p}(\Omega)$ in (3.59) which gives

$$\begin{aligned}
&-\|\nabla u_n^+\|_p^p+\int\limits_{\{u_n^+\le u_\vartheta\}}\left[u_\vartheta^{-\gamma}+\lambda f(x,u_\vartheta)\right]u_n^+dx\\
&+\int\limits_{\{u_\vartheta<u_n^+\}}\left[\left(u_n^+\right)^{-\gamma}+\lambda f(x,u_n^+)\right]u_n^+dx\le\varepsilon_n
\end{aligned}$$

for all $n\in\mathbb{N}$. This leads to

$$-\|\nabla u_n^+\|_p^p+\int\limits_\Omega\left(u_n^+\right)^{1-\gamma}dx+\lambda\int\limits_\Omega f(x,u_n^+)u_n^+dx\le M_6 \tag{3.62}$$

for some $M_6>0$ and for all $n\in\mathbb{N}$. Adding (3.61) and (3.62) yields

$$\int\limits_\Omega \hat{\eta}_\lambda(x,u_n^+)dx\le M_7\quad\text{for some } M_7>0 \text{ and for all } n\in\mathbb{N}. \tag{3.63}$$

Applying (3.63) and reasoning as in the proof of Proposition 3.7 (see the part of the proof after (3.33)), we show that $\{u_n^+\}_{n\ge1}\subseteq W_0^{1,p}(\Omega)$ is bounded and so, due to (3.60), $\{u_n\}_{n\ge1}\subseteq W_0^{1,p}(\Omega)$ is bounded as well.

So, we may assume that

$$u_n\overset{\mathrm{w}}{\to}u\quad\text{in } W_0^{1,p}(\Omega)\quad\text{and}\quad u_n\to u\quad\text{in } L^r(\Omega). \tag{3.64}$$

Choosing $h=u_n-u\in W_0^{1,p}(\Omega)$, passing to the limit as $n\to\infty$ and applying (3.64), we obtain

$$\lim_{n\to\infty}\langle A(u_n),u_n-u\rangle=0,$$

which by the $(S)_+$-property of A, see Proposition 2.2, results in $u_n \to u$ in $W_0^{1,p}(\Omega)$. Therefore, σ_λ satisfies the C-condition and this proves the Claim.

On account of (3.55), (3.56) and the Claim, we are able to apply the mountain pass theorem stated as Theorem 2.1 and find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\sigma_\lambda} \subseteq [u_\vartheta) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad m_\lambda \leq \sigma_\lambda\left(\hat{u}\right), \tag{3.65}$$

see (3.52). From (3.49), (3.55) and (3.65) we conclude that $\hat{u} \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\hat{u} \neq u_0$. This finishes the proof. □

Summarizing the situation for the positive solution of problem (P_λ) as the parameter $\lambda > 0$ varies, we can state the following bifurcation-type theorem.

Theorem 3.9. *If hypotheses H hold, then there exist $\lambda^* > 0$ such that the following is satisfied:*

(a) *problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for all $\lambda \in (0, \lambda^*)$;*
(b) *problem (P_λ) has at least one positive solution $u^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for $\lambda = \lambda^*$;*
(c) *problem (P_λ) has no positive solution for all $\lambda > \lambda^*$.*

4. Minimal positive solutions

In this section we show that problem (P_λ) has a smallest positive solution $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for every $\lambda \in \mathcal{L} = (0, \lambda^*]$ and we prove the monotonicity and continuity properties of the map $\lambda \to \overline{u}_\lambda$.

From Filippakis–Papageorgiou [2] we know that the solution set $\mathcal{S}_\lambda$ is downward directed for every $\lambda \in \mathcal{L} = (0, \lambda^*]$, that is, if $u_1, u_2 \in \mathcal{S}_\lambda$, then there exists $u \in \mathcal{S}_\lambda$ such that $u \leq u_1$ and $u \leq u_2$.

Proposition 4.1. *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem ($P_\lambda$) has a smallest positive solution $\overline{u}_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, that is, $\overline{u}_\lambda \leq u$ for all $u \in \mathcal{S}_\lambda$.*

Proof. Invoking Lemma 3.10 of Hu–Papageorgiou [8, p. 178] we know that there exists a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq \mathcal{S}_\lambda$ such that $\inf \mathcal{S}_\lambda = \inf_{n\geq 1} u_n$. Recall that $\mathcal{S}_\lambda$ is downward directed.

Claim: $\tilde{u} \leq u_n$ for all $n \in \mathbb{N}$ (see the proof of Proposition 3.1).

Fix $n \in \mathbb{N}$ and let $\vartheta \in (0, \lambda) \subseteq \mathcal{L}$. According to Proposition 3.5 there exists $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_n - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. We introduce the Carathéodory function $e_n : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$e_n(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} & \text{if } s < u_\vartheta(x), \\ s^{-\gamma} & \text{if } u_\vartheta(x) \leq s \leq u_n(x), \\ u_n(x)^{-\gamma} & \text{if } u_n(x) < s. \end{cases} \tag{4.1}$$

We set $E_n(x,s) = \int_0^s e_n(x,t)dt$ and consider the C^1-functional $\gamma_n : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_n(u) = \frac{1}{p}\|\nabla u_n\|_p^p - \int_\Omega E_n(x,u)dx.$$

From (4.1) it is clear that γ_n is coercive and the Sobolev embedding theorem implies that γ_n is sequentially weakly lower semicontinuous. Therefore, we find $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\gamma_n(\tilde{u}_0) = \inf\left[\gamma_n(u) : u \in W_0^{1,p}(\Omega)\right].$$

In particular, we have $\gamma_n'(\tilde{u}_0) = 0$ which says that

$$\langle A(\tilde{u}_0), h\rangle = \int_\Omega e_n(x,\tilde{u}_0)h dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{4.2}$$

We choose $h = (u_\vartheta - \tilde{u}_0)^+ \in W_0^{1,p}(\Omega)$ in (4.2). Then, applying (4.1), the nonnegativity of f and the fact that $u_\vartheta \in \mathcal{S}_\vartheta$ gives

$$\begin{aligned}\left\langle A(\tilde{u}_0), (u_\vartheta - \tilde{u}_0)^+\right\rangle &= \int_\Omega u_\vartheta^{-\gamma}(u_\vartheta - \tilde{u}_0)^+ dx\\ &\le \int_\Omega \left[u_\vartheta^{-\gamma} + \vartheta f(x,u_\vartheta)\right](u_\vartheta - \tilde{u}_0)^+ dx\\ &= \left\langle A(u_\vartheta), (u_\vartheta - \tilde{u}_0)^+\right\rangle.\end{aligned}$$

Proposition 2.2 then implies $u_\vartheta \le \tilde{u}_0$. In the same way, choosing $h = (\tilde{u}_0 - u_n)^+ \in W_0^{1,p}(\Omega)$ in (4.2) and applying again (4.1), $f \ge 0$ and $u_n \in \mathcal{S}_\lambda$ results in

$$\begin{aligned}\left\langle A(\tilde{u}_0), (\tilde{u}_0 - u_n)^+\right\rangle &= \int_\Omega u_n^{-\gamma}(\tilde{u}_0 - u_n)^+ dx\\ &\le \int_\Omega \left[u_n^{-\gamma} + \lambda f(x,u_n)\right](\tilde{u}_0 - u_n)^+ dx\\ &= \left\langle A(u_n), (\tilde{u}_0 - u_n)^+\right\rangle.\end{aligned}$$

As before, by Proposition 2.2, we obtain $\tilde{u}_0 \le u_n$. So, we have proved that

$$\tilde{u}_0 \in [u_\vartheta, u_n]. \tag{4.3}$$

From (4.1) and (4.3) it follows that $\tilde{u}_0$ is a positive solution of the auxiliary problem (3.1). Therefore, $\tilde{u}_0 = \tilde{u}$ which implies $\tilde{u} \le u_n$ for all $n \in \mathbb{N}$. This proves the Claim.

We have

$$\langle A(u_n), h\rangle = \int_\Omega \left[u_n^{-\gamma} + \lambda f(x,u_n)\right] h dx \tag{4.4}$$

for all $h \in W_0^{1,p}(\Omega)$ and for all $n \in \mathbb{N}$. Since $0 \leq u_n \leq u_1$ for all $n \geq 1$, from (4.4) with $h = u_n \in W_0^{1,p}(\Omega)$ and using hypothesis H(iv), we infer that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} \overline{u}_\lambda \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to \overline{u}_\lambda \quad \text{in } L^p(\Omega). \tag{4.5}$$

Moreover, we can say that

$$u_n(x)^{-\gamma} \to \overline{u}_\lambda(x)^{-\gamma} \quad \text{for a.a. } x \in \Omega.$$

From the Claim we know that

$$0 \leq u_n(x)^{-\gamma} \leq \tilde{u}(x)^{-\gamma} \quad \text{for a.a. } x \in \Omega.$$

Since $\tilde{u}(\cdot)^{-\gamma} \in L^{p'}(\Omega)$, see the proof of Proposition 3.1, from Gasiński–Papageorgiou [4, Problem 1.19, p. 38], we have

$$u_n^{-\gamma} \xrightarrow{w} \overline{u}_\lambda^{-\gamma} \quad \text{in } L^{p'}(\Omega). \tag{4.6}$$

Therefore, if we choose $h = u_n - \overline{u}_\lambda \in W_0^{1,p}(\Omega)$ in (4.4), pass to the limit as $n \to \infty$ and use (4.5) as well as (4.6), then

$$\lim_{n\to\infty} \langle A(u_n), u_n - \overline{u}_\lambda \rangle = 0,$$

which again by Proposition 2.2 leads to

$$u_n \to \overline{u}_\lambda \quad \text{in } W_0^{1,p}(\Omega). \tag{4.7}$$

So, if we pass to the limit in (4.4) as $n \to \infty$ and use (4.5), (4.6), (4.7), we obtain

$$\langle A(\overline{u}_\lambda), h \rangle = \int_\Omega \left[\overline{u}_\lambda^{-\gamma} + \lambda f(x, \overline{u}_\lambda) \right] h dx \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

From the Claim it follows that $\tilde{u} \leq \overline{u}_\lambda$. Therefore we conclude that

$$\overline{u}_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+ \right) \quad \text{and} \quad \overline{u}_\lambda = \inf \mathcal{S}_\lambda. \quad \square$$

In the next proposition we examine the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ and determine the monotonicity and continuity properties of this map.

Proposition 4.2. *If hypotheses H hold, then the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is*

(a) *strictly increasing, that is,*

$$0 < \vartheta < \lambda \le \lambda^* \quad \textit{implies} \quad \overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right);$$

(b) *left continuous.*

Proof. (a) From Proposition 3.5 we know that there exists $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $\overline{u}_\lambda - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so, since $\overline{u}_\vartheta \le u_\vartheta$, it follows $\overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. So, the map $\lambda \to \overline{u}_\lambda$ is strictly increasing.

(b) Suppose that $\{\lambda_n, \lambda\}_{n\ge1} \subseteq \mathcal{L} = (0, \lambda^*]$ and assume that $\lambda_n \to \lambda^-$. We set $\overline{u}_n = \overline{u}_{\lambda_n} \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for all $n \in \mathbb{N}$. We have

$$\langle A(\overline{u}_n), h\rangle = \int_\Omega \left[\overline{u}_n^{-\gamma} + \lambda_n f(x, \overline{u}_n)\right] h dx \tag{4.8}$$

for all $h \in W_0^{1,p}(\Omega)$ and for all $n \in \mathbb{N}$. Moreover, by Proposition 4.1,

$$0 \le \overline{u}_1 \le \overline{u}_n \le \overline{u}_{\lambda^*}. \tag{4.9}$$

On account of (4.9) and by the choice $h = \overline{u}_n \in W_0^{1,p}(\Omega)$ in (4.8), we infer that $\{\overline{u}_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ is bounded. We have

$$\begin{aligned} -\Delta_p \overline{u}_n &= \overline{u}_n^{-\gamma} + \lambda_n f(x, u_n) \quad && \text{in } \Omega,\\ \overline{u}_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for all $n \in \mathbb{N}$. From (4.9) we see that

$$0 \le \overline{u}_n^{-\gamma} \le \overline{u}_1^{-\gamma} \in L^s(\Omega) \text{ with } s > N \text{ and for all } n \in \mathbb{N},$$

see also H(i). Similarly, (4.9) and hypothesis H(i) imply that

$$\{f(\cdot, \overline{u}_n(\cdot))\}_{n\ge1} \subseteq L^s(\Omega) \text{ is bounded.}$$

Then Proposition 1.3 of Guedda–Véron [7] implies that

$$\|\overline{u}_n\|_\infty \le M_8 \quad \text{for some } M_8 > 0 \text{ and for all } n \in \mathbb{N}.$$

From this as in the proof of Proposition 3.1 and using Theorem 2.1 of Lieberman [10], there exist $\alpha \in (0, 1)$ and $M_9 > 0$ such that

$$\overline{u}_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|\overline{u}_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le M_9 \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

Then, (4.10), the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and the monotonicity of the sequence $\{\overline{u}_n\}_{n\geq 1}$ imply that

$$\overline{u}_n \to \tilde{u}_\lambda \quad \text{in } C_0^1(\overline{\Omega}).$$

We claim that $\tilde{u}_\lambda = \overline{u}_\lambda$. If this is not the case, we can find $z_0 \in \Omega$ such that $\overline{u}_\lambda(z_0) < \tilde{u}_\lambda(z_0)$ which implies $\overline{u}_\lambda(z_0) < \overline{u}_n(z_0)$ for all $n \geq n_0$. But this contradicts (a). Therefore, $\tilde{u}_\lambda = \overline{u}_\lambda$ and so $\lambda \to \overline{u}_\lambda$ is left continuous. □

Summarizing the situation concerning the minimal positive solution of problem (P_λ), we can state the following theorem.

Theorem 4.3. *If hypotheses H hold and* $\lambda \in \mathcal{L} = (0, \lambda^*]$, *then problem* ($P_\lambda$) *has a smallest positive solution* $\overline{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *and the map* $\lambda \to \overline{u}_\lambda$ *from* $\mathcal{L} = (0, \lambda^*]$ *into* $C_0^1(\overline{\Omega})$ *is*

- *strictly increasing, that is,* $0 < \vartheta < \lambda \leq \lambda^*$ *implies* $\overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$;
- *left continuous.*

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