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MULTIPLICITY RESULTS FOR A CLASS OF ELLIPTIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITION

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Abstract. This paper provides multiplicity results for a class of nonlinear elliptic problems under a nonhomogeneous Neumann boundary condition. We prove the existence of three nontrivial solutions to these problems which depend on the Fučík spectrum of the negative p-Laplacian with a Robin boundary condition. Using variational and topological arguments combined with an equivalent norm on the Sobolev space $W^{1,p}$ it is obtained a smallest positive solution, a greatest negative solution, and a sign-changing solution.

1. **Introduction.** The purpose of this article is to investigate the existence and multiplicity of weak solutions to elliptic equations with nonhomogeneous Neumann boundary condition. Specifically, given a bounded domain $\Omega \subseteq \mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and let $1 < p < \infty$, we consider the problem

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} + f(x,u) && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= h(x,u) - \theta |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the negative p-Laplacian, $\partial u/\partial\nu$ denotes the outer normal derivative of u while the values a, b and θ are real parameters specified later. The terms $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$ stand for the positive and negative part of u, respectively, and the perturbations, namely $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, are some Carathéodory functions satisfying suitable hypotheses, see (H) below. For the sake of simplicity we omit the denotation for the trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is applied to the functions on the boundary $\partial\Omega$.

The main goal of this article is to prove the existence of three nontrivial weak solutions of the nonhomogeneous Neumann boundary value problem given in (1.1). More precisely, we establish two extremal constant-sign solutions, namely a smallest positive solution u_+ as well as a greatest negative solution u_-, and finally, the existence of a nontrivial sign-changing solution u_0 lying between these extremal constant-sign solutions is pointed out.

Throughout the paper we impose the following assumptions.

(H) Let $\theta > 0$ be a fixed constant and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying the subsequent conditions:

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(H1) f is bounded on bounded sets.

(H2)

$$\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{p-2}s} = -\infty, \quad \text{uniformly with respect to a.a. } x \in \Omega.$$

(H3)

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = 0, \quad \text{uniformly with respect to a.a. } x \in \Omega.$$

(H4) h is bounded on bounded sets.

(H5) There exists a number $s_\theta > 0$ such that

$$\frac{h(x,s)}{|s|^{p-2}s} < \theta, \qquad \text{for a.a. } x \in \partial\Omega \text{ and for all } |s| > s_\theta.$$

(H6)

$$\lim_{s\to 0} \frac{h(x,s)}{|s|^{p-2}s} = 0, \quad \text{uniformly with respect to a.a. } x \in \partial\Omega.$$

(H7) h satisfies the condition

$$|h(x_1,s_1) - h(x_2,s_2)| \le L\Big[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\Big],$$

for all pairs $(x_1,s_1),(x_2,s_2)$ in $\partial\Omega \times [-K,K]$, where K is a positive constant and $\alpha \in (0,1]$.

By means of the hypotheses (H3) and (H6) we see at once that $f(x,0) = h(x,0) = 0$ reasoning that $u \equiv 0$ is a trivial solution of (1.1). The condition (H7) is a Hölder continuity assumption which is needed to make use of the $C^{1,\alpha}$-regularity of Lieberman (see [20]).

In a recent work of the author [32] there are shown multiplicity results to equations of the form

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where the solutions of (1.2) depend on the so-called Steklov Fučík spectrum of the negative p-Laplacian which was intensively treated by Martínez and Rossi in [22]. The novelty of this paper is on the one hand that the solutions of (1.1) depend on the Robin Fučík spectrum of $-\Delta_p$ (see Section 2 for a detailed introduction) and on the other hand we could drop a hypothesis on the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, which was required in [32], namely

(A1) There exists a number $\delta_f > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s} \ge 0$ for all $0 < |s| \le \delta_f$ and for a.a. $x \in \Omega$.

Assumption (A1) means that the function f must change sign near zero. Now, we do not need this condition on f. Further, regarding the behaviour at infinity, the boundary function in [32] has to satisfy the condition

(A2) $\lim_{|s|\to\infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$ uniformly with respect to a.a. $x \in \partial\Omega$.

We point out that we can replace (A2) by the weaker condition (H5).

Another novelty is the usage of an equivalent norm on the space $W^{1,p}(\Omega)$ obtained by Deng (see [12]) which contains the norm $\|\cdot\|_{L^p(\partial\Omega)}$ instead of $\|\cdot\|_{L^p(\Omega)}$. This ensures, in particular, that suitable energy functionals concerning problem (1.1) (involving appropriate truncation functions to make sure the finiteness of the integrals) satisfy the coercivity and the Palais-Smale condition which is required in our approach. It should be mentioned that we do not need differentiability, polynomial growth, or some integral conditions on the mappings f and h. In order to prove our main results we make use of variational and topological tools, e.g. critical point theory, the mountain-pass theorem, the second deformation lemma and the so-called Robin Fučík spectrum of the negative p-Laplacian.

Elliptic equations with a nonhomogeneous Neumann boundary condition regarding existence and multiplicity of solutions were studied by a number of authors in the last years. Without guarantee of completeness we refer to the papers in [1], [13], [14], [15], [19], [21], [23], [29], [35], and the references therein. With reference to homogeneous Neumann problems, multiple solution results can be found for example in [2], [4], [5] and [6]. In the Dirichlet case there also exists a number of publications according to the subject of multiplicity results, see e.g. in [9], [10], [11], and [17].

The paper is organized as follows. In Section 2 we give the basic notations including the definition of a sub- and supersolution of (1.1), we point out some recent results with regard to the Robin Fučík spectrum of the negative p-Laplacian and we consider a second auxiliary problem which is needed in our treatment. The third section is devoted to the proofs of the existence of specific sub- and supersolutions of (1.1) which leads to the existence of two ordered pairs of sub- and supersolution, one with positive sign and the other one with negative sign. Then, we can derive the existence of two constant-sign solutions thanks to the method of sub- and supersolution dealt in [7]. The existence of extremal constant-sign solutions, more exact a smallest positive solution and a greatest negative solution of (1.1), is shown in Section 4 using functional analytical arguments in association with the properties of the Robin Fučík spectrum of $-\Delta_p$. In the last section we prove the existence of a sign-changing solution applying the fact that every nontrivial solution between the obtained extremal constant-sign solution must be a sign-changing solution provided it is unequal to these extremal solutions. Variational and topological tools like the mountain-pass theorem, critical point theory and the second deformation lemma are found a use in this last section.

2. **Preliminaries.**

By $L^p(\Omega), L^p(\partial\Omega)$ and $W^{1,p}(\Omega)$ we denote the usual Lebesgue and Sobolev spaces with their norms $\|\cdot\|_{L^p(\Omega)}, \|\cdot\|_{L^p(\partial\Omega)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$, respectively. Given $\zeta > 0$ we introduce an equivalent norm on $W^{1,p}(\Omega)$ given by

$$\|u\|_\zeta = \|\nabla u\|_{L^p(\Omega)} + \zeta \|u\|_{L^p(\partial\Omega)} \tag{2.1}$$

(see e.g. Deng [12]). We say that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (1.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx$$
$$= \int_\Omega (a(u^+)^{p-1} - b(u^-)^{p-1} + f(x,u))v dx + \int_{\partial\Omega} (h(x,u) - \theta |u|^{p-2}u) v d\mu,$$

holds for all test functions $v \in W^{1,p}(\Omega)$ while $d\mu$ denotes the usual $(N-1)$-dimensional surface measure. Further, the definition of weak sub- and supersolutions is required in our treatments. A function $\underline{u} \in W^{1,p}(\Omega)$ is said to be a weak subsolution of problem (1.1) if the inequality

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\cdot \nabla v dx \leq \int_\Omega (a(\underline{u}^+)^{p-1} - b(\underline{u}^-)^{p-1} + f(x,\underline{u}))v dx + \int_{\partial\Omega} (h(x,\underline{u}) - \theta|\underline{u}|^{p-2}\underline{u})v d\mu,$$

is satisfied for all nonnegative test functions $v \in W^{1,p}(\Omega)$. Analogously, replacing '$\underline{u}$' by '$\overline{u}$' and '$\leq$' by '$\geq$', we obtain the definition of a weak supersolution of problem (1.1). It is obvious that every weak solution is both a weak subsolution and a weak supersolution. As a consequence of the assumptions in (H) we get a helpful result stated above.

Corollary 2.1. *Under the hypothesis (H) for each $\xi > 0$ there exist constants $\chi_1, \chi_2 > 0$ such that, for all $0 \leq |s| \leq \xi$,*

$$|f(x,s)| \leq \chi_1 |s|^{p-1}, \text{ for a.a. } x \in \Omega, \quad |h(x,s)| \leq \chi_2 |s|^{p-1}, \text{ for a.a. } x \in \partial\Omega.$$

To be more precise, the growth conditions in Corollary 2.1 come from the assumptions (H1), (H3), (H4) and (H6), respectively.

As mentioned in the Introduction, we need the properties of the Fučík spectrum of the negative p-Laplacian $-\Delta_p$ with Robin boundary condition. This spectrum is defined as the set $\widehat{\Sigma}_p$ of all pairs $(a,b) \in \mathbb{R}^2$ such that

$$\begin{aligned} -\Delta_p u &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= -\beta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

is solved nontrivially meaning that $u \in W^{1,p}(\Omega), u \not\equiv 0$, and verifies the equality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx + \beta \int_{\partial\Omega} |u|^{p-2}u v d\mu = \int_\Omega (a(u^+)^{p-1} - b(u^-)^{p-1})v dx, \tag{2.3}$$

for all $v \in W^{1,p}(\Omega)$. In (2.2), respectively (2.3), the parameter β is supposed to be a fixed, nonnegative constant. If $\beta = 0$, (2.2) reduces to the Fučík spectrum Θ_p of the negative Neumann p-Laplacian (see [3]). The special case $a = b = \lambda$ leads to

$$-\Delta_p u = \lambda|u|^{p-2}u \quad \text{in } \Omega, \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = -\beta|u|^{p-2}u \quad \text{on } \partial\Omega, \tag{2.4}$$

which is known as the Robin eigenvalue problem of the negative p-Laplacian. Problem (2.4) was studied in the important publication of Lê [18] devoted to the eigenvalue problems for the negative p-Laplacian. In the Robin case he proved that the first eigenvalue λ_1 of (2.4) corresponding to the fixed value β is simple, isolated and it can be variationally characterized through

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\mu : \int_\Omega |u|^p dx = 1 \right\}. \tag{2.5}$$

Moreover, the set of eigenvalues to (2.4) is closed (see [18, Theorem 5.9]). It is also known that the first eigenfunction φ_1 associated to λ_1 has constant sign in Ω and every eigenfunction corresponding to an eigenvalue greater than λ_1 has to change sign. As $\varphi_1 > 0$ in $\overline{\Omega}$ and φ_1 belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ it follows that $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ where $\operatorname{int}(C^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) \geq 0, \forall x \in \Omega\}$ in the Banach space $C^1(\overline{\Omega})$, which is nonempty and given by

$$\operatorname{int}(C^1(\overline{\Omega})_+) = \left\{u \in C^1(\overline{\Omega}) : u(x) > 0, \forall x \in \overline{\Omega}\right\}.$$

Remark 2.2. *If $\lambda_1^{(\beta)}$ is the first eigenvalue of the Robin eigenvalue problem (2.4) corresponding to the fixed value $\beta > 0$ and if θ is a real parameter satisfying $0 < \theta < \beta$, then we see from (2.5) that $\lambda_1^{(\beta)} \geq \lambda_1^{(\theta)}$ where $\lambda_1^{(\theta)}$ is the first eigenvalue of (2.4) concerning the value θ. This note is required in the end of the proof of Theorem 4.1 demonstrating the existence of extremal constant-sign solutions of (1.1).*

Let us turn back to the Robin Fučík spectrum which was recently studied in [24] through a variational approach using a mountain-pass procedure. More precisely, it was shown that $\widehat{\Sigma}_p$ contains a first nontrivial curve $\mathcal{C}$ which can be expressed as

$$\mathcal{C} = \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\}, \tag{2.6}$$

where $c(s)$ is given by

$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,+1]} \tilde{J}_s(u),$$

with

$$\Gamma = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\}. \tag{2.7}$$

Here, $\tilde{J}_s$ is equal to the restriction of the C^1-functional $J_s : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_s(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\mu - s \int_\Omega (u^+)^p dx$$

to the C^1-submanifold

$$S = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega |u|^p dx = 1 \right\} \tag{2.8}$$

of $W^{1,p}(\Omega)$. This first nontrivial curve stated in (2.6) is Lipschitz continuous, decreasing and its asymptotic behavior can be described by

$$\lim_{s \to +\infty} c(s) = \begin{cases} \lambda_1 & \text{if } p \leq N \\ \overline{\lambda} > \lambda_1 & \text{if } p > N \end{cases}$$

where

$$\overline{\lambda} = \inf_{u \in L} \max_{r \in \mathbb{R}} \frac{\int_\Omega |\nabla(r\varphi_1 + u)|^p dx + \beta \int_{\partial\Omega} |r\varphi_1 + u|^p d\mu}{\int_\Omega |r\varphi_1 + u|^p dx},$$

with

$$L = \{u \in W^{1,p}(\Omega) : u \text{ vanishes somewhere in } \overline{\Omega}, u \not\equiv 0\}$$

(see [24, Proposition 4.2 and Theorem 4.3]). With the help of this first nontrivial curve, we can formulate our last hypothesis on the given data in (1.1).

(H8) Let β be chosen such that $0 < \theta < \beta$ and let $(a, b) \in \mathbb{R}_+^2$ be above the first nontrivial curve $\mathcal{C}$ of the Fučík spectrum $\widehat{\Sigma}_p$ constructed in [24].

In case $a = b = \lambda$ condition (H8) reduces to the assumption that the value λ is strictly greater than λ_2 being the second eigenvalue of the Robin eigenvalue problem of $-\Delta_p$ because of the fact that the point (λ_2, λ_2) belongs to $\widehat{\Sigma}_p$. In [24] a variational characterization of this eigenvalue is obtained by the representation

$$\lambda_2 = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} \left[\int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\sigma \right].$$

For a detailed summary about the Fučík spectrum of the negative p-Laplacian with different boundary conditions we refer to the recent overview article in [25].

A second problem which plays an important part in our treatment is the subsequent boundary value problem

$$-\Delta_p u = -|u|^{p-2}u + 1 \quad \text{in } \Omega, \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 1 \quad \text{on } \partial\Omega, \tag{2.9}$$

which means that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx = -\int_\Omega (|u|^{p-2}u - 1)v dx + \int_{\partial\Omega} v d\mu \tag{2.10}$$

is fulfilled for every test function $v \in W^{1,p}(\Omega)$. From the classical existence theory we infer the existence of a weak solution of problem (2.9). Testing (2.10) with $v = e_1 - e_2$, where $e_1, e_2 \in W^{1,p}(\Omega)$ are two weak solutions of (2.10), we get the uniqueness. Denote by $e \in W^{1,p}(\Omega)$ the unique weak solution of (2.9), we see at once that e must be nonnegative (testing with $v = e^-$). Further, we obtain $e \in L^\infty(\Omega)$ (see [31, Theorem 4.1] or [33, Corollary 1.2]) and from the regularity results of Lieberman [20] it follows $e \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$. Taking (2.9) into account we have

$$\Delta_p e = |e|^{p-2}e - 1 \le e^{p-1} \quad \text{a.e. in } \Omega.$$

Defining $\beta : [0,\infty) \to \mathbb{R}$ through $\beta(s) = s^{p-1}$ for $s > 0$ we may apply Vázquez's strong maximum principle (see [27, Theorem 5]) to get $e(x) > 0$ for all $x \in \Omega$. Fixing $x_0 \in \partial\Omega$ such that $e(x_0) = 0$ and using again Vázquez's strong maximum principle we conclude that $\partial u/\partial\nu(x_0) < 0$. From the boundary condition in (2.9) we obtain $|\nabla u|^{p-2}\partial u/\partial \nu(x_0) = 1$ which is a contradiction. Hence, $e(x) > 0$ in $\overline{\Omega}$ guaranteeing $e \in \text{int}(C^1(\overline{\Omega})_+)$.

3. **Existence of sub- and supersolutions.** In this section we provide the existence of some pairs of weak sub- and supersolutions of our problem (1.1). Here and in the rest of the paper we denote by φ_1 the first eigenfunction of the Robin eigenvalue problem (2.4) corresponding to the first eigenvalue λ_1 related to the fixed parameter β. The function e stands for the unique weak solution of problem (2.9). The main result in this section is the following.

Lemma 3.1. *Let the assumptions in (H) be satisfied and suppose that $a, b > \lambda_1$ as well as $0 < \theta < \beta$. Then there are constants $\vartheta_a, \vartheta_b > 0$ depending on a and b, respectively, such that $\vartheta_a e$ is a positive weak supersolution and $-\vartheta_b e$ is a negative weak subsolution of problem (1.1). Additionally, the function $\varepsilon\varphi_1$ is a positive weak subsolution of problem (1.1) while $-\varepsilon\varphi_1$ is a negative weak supersolution provided the number $\varepsilon > 0$ is sufficiently small.*

Proof. We start to show that $\vartheta_a e$ is a positive weak supersolution of (1.1) with a positive constant ϑ_a to be specified. From (2.10) we obtain

$$\begin{aligned} &\int_\Omega |\nabla(\vartheta_a e)|^{p-2}\nabla(\vartheta_a e) \cdot \nabla v dx \\ &= -\int_\Omega (\vartheta_a e)^{p-1} v dx + \int_\Omega \vartheta_a^{p-1} v dx + \int_{\partial\Omega} \vartheta_a^{p-1} v d\mu, \quad \forall v \in W^{1,p}(\Omega). \end{aligned} \tag{3.1}$$

Combining the definition of a weak supersolution and equation (3.1), we have to show that the inequality

$$\begin{aligned}&\int_\Omega (\vartheta_a^{p-1} - (1+a)(\vartheta_a e)^{p-1} - f(x, \vartheta_a e))v dx \\ &\quad + \int_{\partial\Omega} (\vartheta_a^{p-1} + \theta(\vartheta_a e)^{p-1} - h(x, \vartheta_a e))v d\mu \geq 0\end{aligned} \tag{3.2}$$

is satisfied for all nonnegative test functions $v \in W^{1,p}(\Omega)$. Thanks to condition (H2) there exists a number $s_a > 0$ such that

$$\frac{f(x,s)}{s^{p-1}} < -(1+a), \qquad \text{for a.a. } x \in \Omega \text{ and all } s > s_a. \tag{3.3}$$

With the aid of (H1), one gets

$$f(x,s) + (1+a)s^{p-1} \leq c_a, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_a] \tag{3.4}$$

with a constant c_a depending on a. Finally, from (3.3) and (3.4) it follows

$$f(x,s) \leq -(1+a)s^{p-1} + c_a, \quad \text{for a.a. } x \in \Omega \text{ and all } s \geq 0. \tag{3.5}$$

From (H5) we obtain the existence of $s_\theta > 0$ such that

$$\frac{h(x,s)}{s^{p-1}} < \theta, \qquad \text{for a.a. } x \in \partial\Omega \text{ and all } s > s_\theta,$$

and condition (H4) yields a constant $c_\theta > 0$ such that

$$h(x,s) \leq c_\theta, \qquad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in [0, s_\theta].$$

Consequently this leads to

$$h(x,s) \leq \theta s^{p-1} + c_\theta, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \geq 0. \tag{3.6}$$

Now we can estimate the integrals in (3.2) using the inequalities in (3.5) and (3.6). It results in

$$\begin{aligned}&\int_\Omega (\vartheta_a^{p-1} - (1+a)(\vartheta_a e)^{p-1} - f(x, \vartheta_a e))v dx \\ &\quad + \int_{\partial\Omega} (\vartheta_a^{p-1} + \theta(\vartheta_a e)^{p-1} - h(x, \vartheta_a e))v d\mu \\ &\geq \int_\Omega (\vartheta_a^{p-1} - (1+a)(\vartheta_a e)^{p-1} + (1+a)(\vartheta_a e)^{p-1} - c_a)v dx \\ &\quad + \int_{\partial\Omega} (\vartheta_a^{p-1} + \theta(\vartheta_a e)^{p-1} - \theta(\vartheta_a e)^{p-1} - c_\theta)v d\mu \\ &= \int_\Omega (\vartheta_a^{p-1} - c_a)v dx + \int_{\partial\Omega} (\vartheta_a^{p-1} - c_\theta)v d\mu,\end{aligned}$$

for all $v \in W^{1,p}(\Omega)_+$. From the choice $\vartheta_a := \max\left\{c_a^{\frac{1}{p-1}}, c_\theta^{\frac{1}{p-1}}\right\}$ we conclude that the function $\overline{u} = \vartheta_a e$ is a positive weak supersolution of our problem (1.1). Following the same pattern one can prove that $\underline{u} = -\vartheta_b e$ is a negative weak subsolution of (1.1).

Let us prove the second part of the lemma. To this end, we consider the weak formulation of the Robin eigenvalue problem of the p-Laplacian multiplied with the parameter $\varepsilon^{p-1} > 0$, namely

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\cdot\nabla v dx\\ &= \int_\Omega \lambda_1(\varepsilon\varphi_1)^{p-1} v dx - \int_{\partial\Omega}\beta(\varepsilon\varphi_1)^{p-1} v d\mu, \quad \forall v \in W^{1,p}(\Omega).\end{aligned}$$

Taking the definition of a weak subsolution into account we have to prove that

$$\begin{aligned}&\int_\Omega ((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} - f(x,\varepsilon\varphi_1)) v dx\\ &\quad + \int_{\partial\Omega}((\theta-\beta)(\varepsilon\varphi_1)^{p-1} - h(x,\varepsilon\varphi_1)) v d\mu \le 0\end{aligned} \tag{3.7}$$

is fulfilled for all $v \in W^{1,p}(\Omega)_+$. Applying the assumptions (H3) and (H6) provides the existence of two numbers $\delta_a > 0$ and $\delta_\theta > 0$ such that

$$\begin{aligned}\frac{|f(x,s)|}{|s|^{p-1}} &< a - \lambda_1, \quad \text{for a.a. } x \in \Omega \text{ and all } 0 < |s| \le \delta_a,\\ \frac{|h(x,s)|}{|s|^{p-1}} &< \beta - \theta, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \le \delta_\theta,\end{aligned} \tag{3.8}$$

due to the fact that $a > \lambda_1$ and $\beta > \theta$. Choosing

$$0 < \varepsilon \le \min\left\{\frac{\delta_a}{\|\varphi_1\|_\infty}, \frac{\delta_\theta}{\|\varphi_1\|_\infty}\right\},$$

where $\|\varphi_1\|_\infty$ stands for the supremum-norm of φ_1, along with (3.8), we obtain from (3.7)

$$\begin{aligned}&\int_\Omega ((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} - f(x,\varepsilon\varphi_1)) v dx\\ &\quad + \int_{\partial\Omega}((\theta-\beta)(\varepsilon\varphi_1)^{p-1} - h(x,\varepsilon\varphi_1)) v d\mu\\ &\le \int_\Omega ((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} + (a-\lambda_1)(\varepsilon\varphi_1)^{p-1}) v dx\\ &\quad + \int_{\partial\Omega}((\theta-\beta)(\varepsilon\varphi_1)^{p-1} + (\beta-\theta)(\varepsilon\varphi_1)^{p-1}) v d\mu\\ &= 0,\end{aligned}$$

which proves the assertion. The existence of a negative weak supersolution $-\varepsilon\varphi_1$ can be shown in a similar way. □

Remark 3.2. *Note that every nontrivial weak solution $u \in [0, \vartheta_a e]$ of (1.1) belongs to $\operatorname{int}(C^1(\overline{\Omega})_+)$. This follows from the $C^{1,\alpha}$-regularity of Lieberman [20] combined with Vázquez's strong maximum principle [27] and the growth properties of f and h given in Corollary 2.1. The same holds true for every nontrivial weak solution $u \in [-\vartheta_b e, 0]$ meaning that u lies in $-\operatorname{int}(C^1(\overline{\Omega})_+)$.*

4. **Extremal constant-sign solutions.** The main result in this section is the following theorem about the existence of extremal constant-sign solutions of problem (1.1).

Theorem 4.1. *Let the conditions in (H) be satisfied and let* $0 < \theta < \beta$. *Then, for every* $a > \lambda_1$ *and* $b \in \mathbb{R}$, *there exists a smallest positive weak solution* $u_+ = u_+(a) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ *of (1.1) in the order interval* $[0, \vartheta_a e]$ *while for every* $b > \lambda_1$ *and* $a \in \mathbb{R}$, *there exists a greatest negative weak solution* $u_- = u_-(b) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ *within* $[-\vartheta_b e, 0]$.

Proof. We only prove the assertion for the smallest positive weak solution, the other case acts in the same way. From Lemma 3.1 we know the existence of a positive weak subsolution $\varepsilon\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and a positive weak supersolution $\vartheta_a e \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Taking $\varepsilon > 0$ small enough such that $\varepsilon\varphi_1 \leq \vartheta_a e$ provides an ordered pair of weak sub- and supersolutions of problem (1.1), namely $[\varepsilon\varphi_1, \vartheta_a e]$. The method of weak sub- and supersolution concerning problems of type (1.1) (see [7]) ensures the existence of a smallest positive weak solution $u_\varepsilon = u_\varepsilon(a)$ of (1.1) lying between $\varepsilon\varphi_1$ and $\vartheta_a e$. Taking into account Remark 3.2 we obtain that $u_\varepsilon \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Therefore, for every positive integer n sufficiently large, there exists a smallest positive weak solution $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.1) satisfying $\frac{1}{n}\varphi_1 \leq u_n \leq \vartheta_a e$. From this we get a sequence (u_n) of smallest positive weak solutions being monotone decreasing. It follows

$$u_n \downarrow u_+ \quad \text{pointwise} \tag{4.1}$$

with a function $u_+ : \Omega \to \mathbb{R}$ belonging to $[0, \vartheta_a e]$.

Let us show that u_+ solves problem (1.1). As $u_n \in [\frac{1}{n}\varphi_1, \vartheta_a e]$, one can easily prove the boundedness of (u_n) in $W^{1,p}(\Omega)$. Thus, there is a weakly convergent subsequence of (u_n) and due to the monotonicity of (u_n) along with the compact embeddings $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ as well as $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, the entire sequence (u_n) has the following convergence properties:

$$\begin{aligned} &u_n \rightharpoonup u_+ \quad \text{in } W^{1,p}(\Omega), \\ &u_n \to u_+ \quad \text{in } L^p(\Omega),\ \text{in } L^p(\partial\Omega),\ \text{for a.a. } x \in \Omega,\ \text{and for a.a. } x \in \partial\Omega. \end{aligned} \tag{4.2}$$

As u_n solves problem (1.1), we have

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx \\ &= \int_\Omega (a u_n^{p-1} + f(x, u_n)) v dx + \int_{\partial\Omega} (h(x, u_n) - \theta u_n^{p-1}) v d\mu, \end{aligned} \tag{4.3}$$

for all $v \in W^{1,p}(\Omega)$. Taking the test function $v = u_n - u_+ \in W^{1,p}(\Omega)$ leads to

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u_+) dx \\ &= \int_\Omega (a u_n^{p-1} + f(x, u_n))(u_n - u_+) dx + \int_{\partial\Omega} (h(x, u_n) - \theta u_n^{p-1})(u_n - u_+) d\mu. \end{aligned}$$

Thanks to the boundedness of f and h in combination with the convergence properties in (4.2) and the uniform boundedness of the sequence (u_n), we get by applying Lebesgue's dominated convergence that

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u_+) dx = 0,$$

which by the (S_+)-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ implies

$$u_n \to u_+ \quad \text{in } W^{1,p}(\Omega). \tag{4.4}$$

The strong convergence in (4.4) along with (H1), (H4) and the uniform boundedness of (u_n) allows us to pass to the limit in (4.3) which ensures that u_+ is in fact a weak solution of (1.1).

Taking into account Remark 3.2 we know that $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ provided $u \not\equiv 0$. Arguing by contradiction, we suppose that $u_+ \equiv 0$ implying that (see (4.1))

$$u_n(x) \downarrow 0 \quad \text{for all } x \in \Omega. \tag{4.5}$$

Setting

$$w_n := \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \quad \text{for all } n,$$

we may suppose that, along a subsequence denoted again by w_n,

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{in } W^{1,p}(\Omega), \\ w_n &\to w \quad \text{in } L^p(\Omega),\ \text{in } L^p(\partial\Omega),\ \text{for a.a. } x \in \Omega,\ \text{for a.a. } x \in \partial\Omega, \end{aligned} \tag{4.6}$$

with some function $w \in W^{1,p}(\Omega)$. Additionally, there exist functions $k_1 \in L^p(\Omega)_+$, $k_2 \in L^p(\partial\Omega)_+$ such that

$$\begin{aligned} |w_n(x)| &\le k_1(x) \quad \text{for a.a. all } x \in \Omega, \\ |w_n(x)| &\le k_2(x) \quad \text{for a.a. all } x \in \partial\Omega. \end{aligned} \tag{4.7}$$

Using the representation $u_n = \|u_n\|_{W^{1,p}(\Omega)} w_n$ we have from (4.3) the variational equation

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla v dx \\ &= \int_\Omega \left(a w_n^{p-1} + \frac{f(x,u_n)}{u_n^{p-1}} w_n^{p-1} \right) v dx + \int_{\partial\Omega} \left(\frac{h(x,u_n)}{u_n^{p-1}} w_n^{p-1} - \theta w_n^{p-1} \right) v d\mu, \end{aligned} \tag{4.8}$$

for all $v \in W^{1,p}(\Omega)$. Particularly, for the choice $v = w_n - w \in W^{1,p}(\Omega)$, one gets

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla (w_n - w) dx \\ &= \int_\Omega \left(a w_n^{p-1} + \frac{f(x,u_n)}{u_n^{p-1}} w_n^{p-1} \right) (w_n - w) dx \\ &+ \int_{\partial\Omega} \left(\frac{h(x,u_n)}{u_n^{p-1}} w_n^{p-1} - \theta w_n^{p-1} \right) (w_n - w) d\mu. \end{aligned} \tag{4.9}$$

Applying Corollary 2.1 with $\xi = \vartheta_a \|e\|_\infty$ there exist constants $c_f, c_h > 0$ such that

$$\begin{aligned} &\frac{|f(x,u_n(x))|}{u_n^{p-1}(x)} w_n^{p-1}(x) |w_n(x) - w(x)| \le c_f k_1(x)^{p-1} (k_1(x) + |w(x)|), \\ &\frac{|h(x,u_n(x))|}{u_n^{p-1}(x)} w_n^{p-1}(x) |w_n(x) - w(x)| \le c_h k_2(x)^{p-1} (k_2(x) + |w(x)|), \end{aligned} \tag{4.10}$$

where (4.7) is also taken into account. As the right-hand sides of (4.10) are in $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, we may apply Lebesgue's dominated convergence theorem, which associated with (4.6) provides

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{u_n^{p-1}} w_n^{p-1} (w_n - w) dx = 0, \\ &\lim_{n\to\infty} \int_{\partial\Omega} \frac{h(x,u_n)}{u_n^{p-1}} w_n^{p-1} (w_n - w) d\mu = 0. \end{aligned} \tag{4.11}$$

From (4.9) in conjunction with (4.11) we derive

$$\limsup_{n\to\infty}\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\cdot\nabla(w_n-w)dx=0.$$

Applying again the (S_+)-property of $-\Delta_p$ corresponding to $W^{1,p}(\Omega)$ yields

$$w_n\to w \quad \text{ in } W^{1,p}(\Omega), \tag{4.12}$$

while $\|w\|_{W^{1,p}(\Omega)}=1$ meaning $w\not\equiv 0$. Taking into account (4.5), (4.12), and the assumptions (H3) and (H6), we may pass to the limit in (4.8) which results in

$$\int_\Omega |\nabla w|^{p-2}\nabla w\cdot\nabla v dx = a\int_\Omega w^{p-1}vdx-\theta\int_{\partial\Omega} w^{p-1}vd\mu, \quad \forall v\in W^{1,p}(\Omega). \tag{4.13}$$

Since $w\not\equiv 0$, equation (4.13) represents the Robin eigenvalue problem of the negative p-Laplacian $-\Delta_p$ with the eigenfunction $w\geq 0$ corresponding to the eigenvalue a and related to the parameter θ. By means of Remark 2.2 and due to the assumptions $a>\lambda_1$ and $0<\theta<\beta$, we see that a is also greater than the first eigenvalue of the Robin eigenvalue problem corresponding to the positive number θ. However, this contradicts the results of Lê [18] because w must change sign in Ω. Hence, $u_+\not\equiv 0$ concluding $u_+\in \operatorname{int}(C^1(\overline{\Omega})_+)$.

Finally, we have to check that u_+ is indeed the smallest positive weak solution in $[0,\vartheta_a e]$. To this end, let $u\in W^{1,p}(\Omega), 0\leq u\leq \vartheta_a e, u\not\equiv 0$ be a weak solution of (1.1). Remark 3.2 ensures that $u\in \operatorname{int}(C^1(\overline{\Omega})_+)$. This implies the existence of an integer n sufficiently large such that $u\in[\frac{1}{n}\varphi_1,\vartheta_a e]$. As we already know, u_n is the smallest weak solution in the ordered interval $[\frac{1}{n}\varphi_1,\vartheta_a e]$ meaning that $u_n\leq u$. Making use of (4.1), we get $u_+\leq u$ which proves that $u_+\in \operatorname{int}(C^1(\overline{\Omega})_+)$ is the smallest weak solution of problem (1.1) within $[0,\vartheta_a e]$. □

Remark 4.2. *Regarding Theorem 4.1 the next proceeding is to find a third nontrivial weak solution u_0 which lies between u_- and u_+. If $u_0\neq u_-$ and $u_0\neq u_+$, then it must be a sign-changing weak solution of (1.1) due to the extremality of u_+ and u_-.*

5. **Sign-changing solution.**

In this section we prove the existence of a nontrivial sign-changing weak solution $u_0\in C^1(\overline{\Omega})$ of (1.1) which belongs to the ordered interval $[u_-,u_+]$.

To this end, let $\tau_+,\tau_-,\tau_0:\Omega\times\mathbb{R}\to\mathbb{R}$ be truncation operators defined as:

$$\tau_+(x,s)=\begin{cases}0 & \text{if } s<0\\ s & \text{if } 0\leq s\leq u_+(x)\\ u_+(x) & \text{if } s>u_+(x)\end{cases}, \quad \tau_-(x,s)=\begin{cases}u_-(x) & \text{if } s<u_-(x)\\ s & \text{if } u_-(x)\leq s\leq 0\\ 0 & \text{if } s>0\end{cases},$$

$$\tau_0(x,s)=\begin{cases}u_-(x) & \text{if } s<u_-(x)\\ s & \text{if } u_-(x)\leq s\leq u_+(x)\\ u_+(x) & \text{if } s>u_+(x)\end{cases}.$$

Further, we denote by $\tau_+^{\partial\Omega},\tau_-^{\partial\Omega},\tau_0^{\partial\Omega}:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ the corresponding truncation operators defined on $\partial\Omega$. We see at once that these truncation functions are continuous, uniformly bounded, and even Lipschitz continuous with respect to the second argument. Taking into account these truncations we introduce the subsequent associated functionals through

$$\begin{aligned}
J_+(u) =& \frac{1}{p}\left(\|\nabla u\|^p_{L^p(\Omega)} + \theta\|u\|^p_{L^p(\partial\Omega)}\right)\\
&-\int_\Omega\int_0^{u(x)}\left(a\tau_+(x,s)^{p-1} + f(x,\tau_+(x,s))\right)dsdx\\
&-\int_{\partial\Omega}\int_0^{u(x)} h(x,\tau_+^{\partial\Omega}(x,s))dsd\mu,\\
J_-(u) =& \frac{1}{p}\left(\|\nabla u\|^p_{L^p(\Omega)} + \theta\|u\|^p_{L^p(\partial\Omega)}\right)\\
&-\int_\Omega\int_0^{u(x)}\left(-b|\tau_-(x,s)|^{p-1} + f(x,\tau_-(x,s))\right)dsdx\\
&-\int_{\partial\Omega}\int_0^{u(x)} h(x,\tau_-^{\partial\Omega}(x,s))dsd\mu,\\
J_0(u) =& \frac{1}{p}\left(\|\nabla u\|^p_{L^p(\Omega)} + \theta\|u\|^p_{L^p(\partial\Omega)}\right)\\
&-\int_\Omega\int_0^{u(x)}\left(a\tau_+(x,s)^{p-1} - b|\tau_-(x,s)|^{p-1} + f(x,\tau_0(x,s))\right)dsdx\\
&-\int_{\partial\Omega}\int_0^{u(x)} h(x,\tau_0^{\partial\Omega}(x,s))dsd\mu.
\end{aligned}$$

These functionals are well-defined and differentiable. Thanks to the truncation operators combined with the equivalent norm stated in (2.1) (replacing ζ by θ) it can be shown that $J_-, J_+, J_0 : W^{1,p}(\Omega) \to \mathbb{R}$ are coercive and weakly sequentially lower semicontinuous implying the existence of their global minimizers (cf. e.g. [34, Proposition 38.15]). Further, the functionals fulfill the classical Palais-Smale condition. A characterization of their critical points is stated in the next lemma.

Lemma 5.1. *Every critical point $\omega \in W^{1,p}(\Omega)$ of $J_+(J_-)$ is a nonnegative (nonpositive) weak solution of (1.1) such that $0 \le \omega \le u_+$ ($u_- \le \omega \le 0$), where u_+ and u_- denote the extremal constant-sign solutions of (1.1) obtained in Theorem 4.1. If $\omega \in W^{1,p}(\Omega)$ is a critical point of J_0, then ω is a weak solution of (1.1) satisfying $u_- \le \omega \le u_+$.*

Proof. Let us show the last assertion, the other ones can be done similarly. Suppose $\omega \in W^{1,p}(\Omega)$ is a critical point of J_0, then it holds $J_0'(\omega) = 0$ meaning that

$$\begin{aligned}
&\int_\Omega |\nabla\omega|^{p-2}\nabla\omega\cdot\nabla v dx + \theta\int_{\partial\Omega}|\omega|^{p-2}\omega v d\mu\\
&=\int_\Omega\left(a\tau_+(x,\omega)^{p-1} - b|\tau_-(x,\omega)|^{p-1} + f(x,\tau_0(x,\omega))\right)vdx\\
&\quad+\int_{\partial\Omega} h(x,\tau_0^{\partial\Omega}(x,\omega))vd\mu, \quad \forall v \in W^{1,p}(\Omega).
\end{aligned}$$

Since u_+ is a positive weak solution of (1.1), we obtain

$$\begin{aligned}&\int_\Omega |\nabla u_+|^{p-2}\nabla u_+ \cdot \nabla v dx\\ &= \int_\Omega \left(au_+^{p-1} + f(x,u_+)\right) v dx + \int_{\partial\Omega}\left(h(x,u_+) - \theta u_+^{p-1}\right) v d\mu,\end{aligned}$$

for all $v \in W^{1,p}(\Omega)$. Now, putting $v = (\omega - u_+)^+$ and combining both equations above, one has

$$\begin{aligned}&\int_\Omega \left(|\nabla\omega|^{p-2}\nabla\omega - |\nabla u_+|^{p-2}\nabla u_+\right)\cdot\nabla(\omega-u_+)^+ dx\\ &\quad + \theta\int_{\partial\Omega}\left(|\omega|^{p-2}\omega - u_+^{p-1}\right)(\omega-u_+)^+ d\mu\\ &= \int_\Omega \left(a\tau_+(x,\omega)^{p-1} - b|\tau_-(x,\omega)|^{p-1} - au_+^{p-1}\right)(\omega-u_+)^+ dx\\ &\quad + \int_\Omega \left(f(x,\tau_0(x,\omega)) - f(x,u_+)\right)(\omega-u_+)^+ dx\\ &\quad + \int_{\partial\Omega}\left(h(x,\tau_0^{\partial\Omega}(x,s)) - h(x,u_+)\right)(\omega-u_+)^+ d\mu.\end{aligned} \tag{5.1}$$

With a view to the definition of the truncation operators it is easy to see that the right-hand side of (5.1) vanishes. However, if $\omega > u_+$, the left-hand side is strictly positive (cf. e.g. [8, p. 37]). Hence, it must hold $\omega \le u_+$. In order to prove $u_- \le \omega$ we can proceed in the same line which yields $u_- \le \omega \le u_+$. Taking again the definition of the truncations into account we have $\tau_+(x,\omega) = \omega^+, |\tau_-(x,\omega)| = \omega^-$, $\tau_0(x,\omega) = \omega$ and $\tau_0^{\partial\Omega}(x,\omega) = \omega$ meaning that ω is a weak solution of (1.1) satisfying $u_- \le \omega \le u_+$. □

Lemma 5.2. *Suppose (H) and let $a, b > \lambda_1$ and $\beta > \theta > 0$. Then the extremal positive (negative) weak solution u_+ (u_-) of (1.1) is the unique global minimizer of the functional J_+ (J_-) while both of them are local minimizers of the functional J_0 as well. Further, J_0 possesses a global minimizer ω_0 being a nontrivial weak solution of (1.1) satisfying $u_- \le \omega_0 \le u_+$.*

Proof. Let $\omega_+ \in W^{1,p}(\Omega)$ the global minimizer of J_+ which exists due to the property of J_+ to be coercive and weakly sequentially lower semicontinuous. Concerning Lemma 5.1 the critical point ω_+ is a nonnegative weak solution to equation (1.1) belonging to $[0, u_+]$. In order to verify that ω_+ is unequal zero, we have to show that $J_+(\omega_+) \neq 0$. According to hypothesis (H3) and (H6) we find numbers $\delta_a > 0$ and $\delta_\theta > 0$ such that

$$\begin{aligned}|f(x,s)| &\le (a-\lambda_1)s^{p-1}, \quad \forall s: 0 < s \le \delta_a,\\ |h(x,s)| &\le (\beta-\theta)s^{p-1}, \quad \forall s: 0 < s \le \delta_\theta,\end{aligned} \tag{5.2}$$

since $a > \lambda_1$ and $\beta > \theta$. We put $\varepsilon > 0$ sufficiently small such that

$$\varepsilon\varphi_1 < u_+, \quad \varepsilon\|\varphi_1\|_\infty < \delta_a, \quad \varepsilon\|\varphi_1\|_\infty < \delta_\theta.$$

From (5.2) combined with the Robin eigenvalue problem we infer

$$\begin{aligned}
J_+(\varepsilon\varphi_1) &= \frac{\lambda_1 - a}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\Omega)} \\
&\quad - \int_\Omega \int_0^{\varepsilon\varphi_1(x)} f(x,s)dsdx + \frac{\theta-\beta}{p}\varepsilon^p\|\varphi_1\|^p_{L^p(\partial\Omega)} - \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} h(x,s)dsd\mu \\
&< \frac{\lambda_1-a}{p}\varepsilon^p\|\varphi_1\|^p_{L^p(\Omega)} + \int_\Omega\int_0^{\varepsilon\varphi_1(x)}(a-\lambda_1)s^{p-1}dsdx + \frac{\theta-\beta}{p}\varepsilon^p\|\varphi_1\|^p_{L^p(\partial\Omega)} \\
&\quad + \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)}(\beta-\theta)s^{p-1}dsd\mu \\
&= 0.
\end{aligned}$$

Hence, $J_+(\omega_+) < 0$ meaning that $\omega_+ \not\equiv 0$. This yields that $\omega_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (cf. Remark 3.2). Due to the fact that u_+ is the smallest positive weak solution of (1.1) in $[0, \vartheta_a e]$ satisfying $0 \le \omega_+ \le u_+$ we obtain $\omega_+ = u_+$ proving that u_+ must be the unique global minimizer of J_+. In a similar way, we get that u_- is the unique global minimizer of J_-. As $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$, there exists a neighborhood V_{u_+} of u_+ in the space $C^1(\overline{\Omega})$ satisfying $V_{u_+} \subset C^1(\overline{\Omega})_+$. Hence, it holds $J_+ = J_0$ on V_{u_+} meaning that u_+ is a local minimizer of J_0 on $C^1(\overline{\Omega})$. From [30] we know that u_+ is a local minimizer on $W^{1,p}(\Omega)$ as well. Similarly, we obtain that u_- is a local minimizer of J_0 with respect to $W^{1,p}(\Omega)$.

As mentioned at the beginning of this section the functional $J_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. That means that its global minimizer, namely ω_0, exists. Taking into account Lemma 5.1 we get that the critical point ω_0 is a solution of (1.1) satisfying $u_- \le \omega_0 \le u_+$. Since $J_0(u_+) = J_+(u_+) < 0$ it follows that ω_0 must be nontrivial meaning $\omega_0 \not\equiv 0$. □

Now, we are in the position to prove the main result in this section.

Theorem 5.3. *Under the hypotheses (H) and (H8) problem (1.1) possesses a nontrivial sign-changing weak solution $u_0 \in C^1(\overline{\Omega})$.*

Proof. In Lemma 5.2 it has been shown that the functional $J_0 : W^{1,p}(\Omega) \to \mathbb{R}$ possesses a global minimizer $\omega_0 \in W^{1,p}(\Omega)$ which is a nontrivial weak solution of our original problem (1.1) lying between u_- and u_+. If $\omega_0 \ne u_-$ and $\omega_0 \ne u_+$, then $u_0 = \omega_0$ must be a sign-changing weak solution of (1.1) due to the extremality properties of the constant-sign solutions u_- and u_+ (cf. Theorem 4.1). In this case we are done.

Let us prove the assertion if either $\omega_0 = u_-$ or $\omega_0 = u_+$ is satisfied. We only show the case $\omega_0 = u_+$, the other one can be done likewise. From Lemma 5.2 it is known that u_- is a local minimizer of J_0. Without loss of generality we can assume that u_- is a strict local minimizer of J_0 elsewise there would exist infinitely many critical points ω of J_0 being sign-changing weak solutions of (1.1) because of the relation $u_- \le \omega \le u_+$ combined with the fact that u_- as well as u_+ are extremal constant-sign solutions. With the aid of these assumptions we find a number $\rho \in \big(0, \|u_+ - u_-\|_{W^{1,p}(\Omega)}\big)$ such that

$$J_0(u_+) \le J_0(u_-) < \inf\{J_0(u) : u \in \partial B_\rho(u_-)\} \tag{5.3}$$

with $\partial B_\rho = \{u \in W^{1,p}(\Omega) : \|u - u_-\|_{W^{1,p}(\Omega)} = \rho\}$. Now, we are able to apply the mountain-pass theorem used to the functional J_0 (see [26] or [28, Theorem 2.4.4]).

Note that J_0 fulfills the Palais-Smale condition which is required at this point. We obtain the existence of a critical point $u_0 \in W^{1,p}(\Omega)$ of J_0, that is $J_0'(u_0) = 0$, satisfying

$$\inf \{J_0(u) : u \in \partial B_\rho(u_-)\} \le J_0(u_0) = \inf_{\gamma \in \Gamma_W} \max_{t \in [-1,1]} J_0(\gamma(t)), \tag{5.4}$$

where

$$\Gamma_W = \{\gamma \in C([-1,1], W^{1,p}(\Omega)) : \gamma(-1) = u_-, \gamma(1) = u_+\}.$$

We easily see from (5.3) and (5.4) that u_0 can not be u_- as well as u_+. Therefore, the critical point u_0 is a nontrivial sign-changing weak solution of (1.1) provided $u_0 \neq 0$. That means we have to prove that $J_0(u_0) \neq 0$, which is satisfied if there exists a path $\widetilde{\gamma} \in \Gamma_W$ such that

$$J_0(\widetilde{\gamma}(t)) \neq 0, \qquad \text{for all } t \in [-1,1]. \tag{5.5}$$

Let S (defined in (2.8)) and $S_C = S \cap C^1(\overline{\Omega})$ be equipped with the topologies induced by $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$, respectively. We set

$$\Gamma_C = \{\gamma \in C([-1,1], S_C) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\}$$

while Γ is stated in (2.7). Taking the recent results in [24] into account there exists a continuous path $\gamma \in \Gamma$ satisfying $t \mapsto \gamma(t) \in \{u \in W^{1,p}(\Omega) : J_{(a,b)}(u) < 0, \|u\|_{L^p(\Omega)} = 1\}$ provided the pair (a,b) is above the curve $\mathcal{C}$ of hypothesis (H8). Here, the functional $J_{(a,b)} : W^{1,p}(\Omega) \to \mathbb{R}$ is defined as the potential associated to the Robin Fučík spectrum given by

$$J_{(a,b)}(u) = \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\mu - \int_\Omega (a(u^+)^p + b(u^-)^p) dx.$$

Thanks to this first nontrivial curve $\mathcal{C}$, we find a number $\kappa > 0$ such that

$$J_{(a,b)}(\gamma(t)) \le -\kappa < 0, \qquad \text{for all } t \in [-1,1].$$

Since S_C is dense in S we deduce the density of Γ_C in Γ (for the proof we refer to [28]) which implies the existence of a continuous path $\gamma_C \in \Gamma_C$ such that

$$|J_{(a,b)}(\gamma(t)) - J_{(a,b)}(\gamma_C(t))| < \frac{\kappa}{2}, \qquad \text{for all } t \in [-1,1].$$

Further, as the set $\gamma_C([-1,1])(\overline{\Omega})$ is uniformly bounded in $\mathbb{R}$ there exists a constant $M > 0$ such that

$$|\gamma_C(t)(x)| \le M \qquad \text{for all } x \in \overline{\Omega} \text{ and for all } t \in [-1,1].$$

Recall that $u_+, -u_- \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (see Theorem 4.1). Then, for every $u \in \gamma_C([-1,1])$ and any bounded neighborhood U_u of u in $C^1(\overline{\Omega})$, we find positive numbers ς_u and ι_u satisfying

$$u_+ - \varsigma w \in \operatorname{int}(C^1(\overline{\Omega})_+) \quad \text{and} \quad -u_- + \iota w \in \operatorname{int}(C^1(\overline{\Omega})_+), \tag{5.6}$$

for all $\varsigma \in [0, \varsigma_u]$, for all $\iota \in [0, \iota_u]$, and for all $w \in U_u$. With the aid of (5.6) together with a compactness argument we obtain the existence of a number $\varepsilon_C > 0$ such that

$$u_-(x) \le \varepsilon \gamma_C(t)(x) \le u_+(x), \tag{5.7}$$

for all $x \in \Omega$, for all $t \in [-1,1]$, and for all $\varepsilon \in (0, \varepsilon_C]$. By means of the representation of $J_{(a,b)}$, we may write the functional J_0 in the subsequent form

$$\begin{aligned} J_0(u) =& \frac{1}{p} J_{(a,b)}(u) + \frac{\theta-\beta}{p} \|u\|^p_{L^p(\partial\Omega)} + \frac{1}{p}\int_\Omega \left(a(u^+)^p + b(u^-)^p\right) dx \\ & - \int_\Omega \int_0^{u(x)} \left(a\tau_+(x,s)^{p-1} - b|\tau_-(x,s)|^{p-1} + f(x,\tau_0(x,s))\right) ds dx \\ & - \int_{\partial\Omega}\int_0^{u(x)} h(x,\tau_0^{\partial\Omega}(x,s)) ds d\mu. \end{aligned} \tag{5.8}$$

Applying (5.7) to the new representation of J_0 given in (5.8) it follows, for all $\varepsilon \in (0,\varepsilon_C]$ and all $t \in [-1,1]$,

$$\begin{aligned} J_0(\varepsilon\gamma_C(t)) &< \frac{1}{p} J_{(a,b)}(\varepsilon\gamma_C(t)) - \int_\Omega \int_0^{\varepsilon\gamma_C(t)(x)} f(x,s) ds dx \\ &\quad - \int_{\partial\Omega}\int_0^{\varepsilon\gamma_C(t)(x)} h(x,s) ds d\mu \\ &= \varepsilon^p \left[\frac{1}{p} J_{(a,b)}(\gamma_C(t)) - \frac{1}{\varepsilon^p}\int_\Omega \int_0^{\varepsilon\gamma_C(t)(x)} f(x,s) ds dx \right. \\ &\qquad \left. - \frac{1}{\varepsilon^p}\int_{\partial\Omega}\int_0^{\varepsilon\gamma_C(t)(x)} h(x,s) ds d\mu \right] \\ &< \varepsilon^p \left[-\frac{\kappa}{2p} + \frac{1}{\varepsilon^p}\int_\Omega \left| \int_0^{\varepsilon\gamma_C(t)(x)} f(x,s) ds \right| dx \right. \\ &\qquad \left. + \frac{1}{\varepsilon^p}\int_{\partial\Omega}\left| \int_0^{\varepsilon\gamma_C(t)(x)} h(x,s) ds \right| d\mu \right], \end{aligned} \tag{5.9}$$

where we have used the fact that $\theta < \beta$. Due to the assumptions (H3) and (H6) there are constants $\psi_1 > 0$ and $\psi_2 > 0$ such that

$$\begin{aligned} |f(x,s)| &\le \frac{\kappa}{5M^p|\Omega|}|s|^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } s : |s| \le \psi_1, \\ |h(x,s)| &\le \frac{\kappa}{5M^p|\partial\Omega|}|s|^{p-1}, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s : |s| \le \psi_2. \end{aligned}$$

Now, we choose $\varepsilon > 0$ sufficiently small such that $\varepsilon < \min\left\{\varepsilon_C, \frac{\psi_1}{M}, \frac{\psi_2}{M}\right\}$ to obtain

$$\begin{aligned} \frac{1}{\varepsilon^p}\int_\Omega \left| \int_0^{\varepsilon\gamma_C(t)(x)} f(x,s) ds \right| dx \le \frac{\kappa}{5p}, \\ \frac{1}{\varepsilon^p}\int_{\partial\Omega} \left| \int_0^{\varepsilon\gamma_C(t)(x)} h(x,s) ds \right| d\mu \le \frac{\kappa}{5p}. \end{aligned} \tag{5.10}$$

Then, we get from (5.9) combined with (5.10)

$$J_0(\varepsilon\gamma_C(t)) \le \varepsilon^p \left(-\frac{\kappa}{2p} + \frac{\kappa}{5p} + \frac{\kappa}{5p} \right) < 0, \qquad \text{for all } t \in [-1,1]. \tag{5.11}$$

Relation (5.11) demonstrates the existence of a continuous path $\varepsilon\gamma_C$ connecting $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. In order to prove the latter in (5.5) we have to construct two other paths which shall join $\varepsilon\varphi_1$ and u_+, respectively, u_- and $-\varepsilon\varphi_1$.

As proved in Lemma 5.2 the smallest positive weak solution u_+ is the unique global minimizer of J_+, so we can suppose that $J_+(u_+) < J_+(\varepsilon\varphi_1)$. Further, from Lemma 5.1 it is known that the functional J_+ has no critical values in the interval $(J_+(u_+), J_+(\varepsilon\varphi_1)]$. Since the functional J_+ satisfies the Palais-Smale condition due to its coercivity, the second deformation lemma (cf. [16]) can be applied to J_+. Denote

$$J_+^{\varepsilon\varphi_1} := \{u \in W^{1,p}(\Omega) : J_+(u) \leq J_+(\varepsilon\varphi_1)\},$$

we obtain the existence of a continuous mapping

$$\eta : [0,1] \times J_+^{\varepsilon\varphi_1} \to J_+^{\varepsilon\varphi_1}$$

characterized through

$$(1) \quad \eta(0,u) = u, \qquad (2) \quad \eta(1,u) = u_+, \qquad (3) \quad J_+(\eta(t,u)) \leq J_+(u),$$

for all $t \in [0,1]$ and for all $u \in J_+^{\varepsilon\varphi_1}$. Now we denote by γ_+ a path from $[0,1]$ to $W^{1,p}(\Omega)$ defined by $\gamma_+(t) = \eta(t,\varepsilon\varphi_1)^+ = \max\{\eta(t,\varepsilon\varphi_1), 0\}$ for all $t \in [0,1]$. Clearly, γ_+ is continuous and joins $\varepsilon\varphi_1$ and u_+. Moreover, it satisfies

$$J_0(\gamma_+(t)) = J_+(\gamma_+(t)) \leq J_+(\eta(t,\varepsilon\varphi_1)) \leq J_+(\varepsilon\varphi_1) < 0,$$

for all $t \in [0,1]$. Finally, we may apply the second deformation lemma to the functional J_- making use of the same arguments. We obtain a continuous path $\gamma_- : [0,1] \to W^{1,p}(\Omega)$ connecting $-\varepsilon\varphi_1$ and u_- and satisfying

$$J_0(\gamma_-(t)) < 0, \qquad \text{for all } t \in [0,1].$$

Now, the proof is almost finished. If we put the paths γ_-, γ_C and γ_+ together, we get a continuous path $\widetilde{\gamma}$ which joins u_- and u_+ and it fulfills (5.5) meaning that $u_0 \in W^{1,p}(\Omega)$ obtained from the mountain-pass theorem is nontrivial. That means that we have found a sign-changing weak solution u_0 of our original problem (1.1) which lies between u_- and u_+. That finishes the proof of the theorem. □

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