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Positive solutions for weighted singular *p*-Laplace equations via Nehari manifolds

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ABSTRACT
In this paper, we study weighted singular p-Laplace equations involving a bounded weight function which can be discontinuous. Due to its discontinuity classical regularity results cannot be applied. Based on Nehari manifolds we prove the existence of at least two positive bounded solutions of such problems.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper, we study the following nonlinear singular Dirichlet problem :

$$\begin{aligned} & -\operatorname{div}(\xi(x)|\nabla u|^{p-2}\nabla u) = a(x)u^{-\gamma} + \lambda u^{r-1} \quad \text{in } \Omega \\ & u\big|_{\partial\Omega} = 0, \quad 0 < \gamma < 1, \quad 1 < p < r < p^*, \quad u \geq 0, \quad \lambda > 0. \end{aligned} \tag{P_λ}$$

In this problem the differential operator is a weighted p-Laplacian with a weight $\xi \in L^\infty(\Omega)$, $\xi \geq 0$ and ξ is supposed to be bounded away from zero. Since ξ is discontinuous in general, we cannot use the nonlinear global regularity theory of Lieberman [1] and the nonlinear strong maximum principle, see Pucci and Serrin [2, p.111 and 120]. The fact that these two basic tools are no longer available leads to a different approach in the analysis of problem (P_λ) which is based on the Nehari method. On the right-hand side of (P_λ) we have the competing effects of two different nonlinearities. One is the singular term $s \to a(x)s^{-\gamma}$ with $s > 0$ and the other one is a parametric $(p-1)$-superlinear perturbation $s \to \lambda s^{r-1}$ with $s \geq 0$ and $p < r < p^*$ with p^* being the critical Sobolev exponent corresponding to p defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$$

We are looking for positive solutions of problem (P_λ) and we show that problem (P_λ) has at least two positive solutions for all $\lambda \geq 0$.

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Singular problems with such competition phenomena were investigated by Sun et al. [3] and Haitao [4] for semilinear equations driven by the Laplacian and by Giacomoni et al. [5], Papageorgiou and Smyrlis [6], Papageorgiou and Winkert [7] and Perera and Zhang [8] for equations driven by the p-Laplacian. We also refer to the works of Leonardi and Papageorgiou [9,10]. In all the mentioned works the weight function ξ is equal to one and so we can use the global elliptic regularity theory and the strong maximum principle. These tools are crucial in the proofs of the works above and are combined with variational methods and suitable truncation and comparison techniques. The regularity theory guarantees that the solutions are in $C_0^1(\bar{\Omega})$ and then the strong maximum principle, so-called Hopf theorem, implies that these solutions are in $\operatorname{int}\left(C_0^1(\bar{\Omega})_+\right)$ which is the interior of the positive order cone of $C_0^1(\bar{\Omega})$.

Without these facts the proofs of the works above are no more valid. As we already indicated, in our setting, these results do not hold, so we need to employ a different approach.

2. Preliminaries

We denote by $W_0^{1,p}(\Omega)$ the usual Sobolev space with norm $\|\cdot\|$. By the Poincaré inequality we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } \; u \in W_0^{1,p}(\Omega),$$

where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$ and $L^p\left(\Omega;\mathbb{R}^N\right)$, respectively. The norm of $\mathbb{R}^N$ is denoted by $|\cdot|$ and '$\cdot$' stands for the inner product in $\mathbb{R}^N$. By $p^* > 1$ we denote the Sobolev critical exponent for p defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \le p. \end{cases}$$

Let $\xi \in L^\infty(\Omega)$ with $0 < \operatorname*{ess\,inf}_{\Omega} \xi$ and let $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ with $(1/p) + (1/p') = 1$ be defined by

$$\langle A(u),\varphi\rangle = \int_\Omega \xi(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x \quad \text{for all } u,\varphi \in W_0^{1,p}(\Omega). \tag{1}$$

The next proposition states the main properties of this map and it can be found in Gasiński and Papageorgiou [11, Problem 2.192, p.279].

Proposition 2.1: *The map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined in* (1) *is bounded, that is, it maps bounded sets to bounded sets, continuous, strictly monotone, hence maximal monotone and it is of type* $(\mathrm{S})_+$, *that is,*

$$u_n \xrightarrow{\mathrm{w}} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$$

imply $u_n \to u$ in $W_0^{1,p}(\Omega)$.

3. Positive solutions

We suppose the following hypotheses related to problem (P_λ) throughout this paper:

$$H_0 : \xi, \quad a \in L^\infty(\Omega), \qquad 0 < \xi_0 \le \operatorname*{ess\,inf}_{\Omega} \xi, \quad a(x) > 0 \text{ for a.a. } x \in \Omega.$$

This hypothesis implies that the natural function space for the analysis of problem (P_λ) is the Sobolev space $W_0^{1,p}(\Omega)$.

Let $\varphi_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\int_\Omega \xi(x)|\nabla u|^p \,dx - \frac{1}{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma}\,dx - \frac{\lambda}{r}\|u\|_r^r.$$

It is clear that φ_λ is not C^1. The corresponding Nehari manifold for this functional is given by

$$N_\lambda = \left\{ u \in W_0^{1,p}(\Omega) \colon \int_\Omega \xi(x)|\nabla u|^p\,dx = \int_\Omega a(x)|u|^{1-\gamma}\,dx + \lambda\|u\|_r^r,\ u \neq 0 \right\}.$$

We decompose N_λ into three disjoint parts

$$\begin{aligned}
N_\lambda^+ &= \left\{ u \in N_\lambda \colon (p+\gamma-1)\int_\Omega \xi(x)|\nabla u|^p\,dx - \lambda(r+\gamma-1)\|u\|_r^r > 0 \right\},\\
N_\lambda^0 &= \left\{ u \in N_\lambda \colon (p+\gamma-1)\int_\Omega \xi(x)|\nabla u|^p\,dx = \lambda(r+\gamma-1)\|u\|_r^r \right\},\\
N_\lambda^- &= \left\{ u \in N_\lambda \colon (p+\gamma-1)\int_\Omega \xi(x)|\nabla u|^p\,dx - \lambda(r+\gamma-1)\|u\|_r^r < 0 \right\}.
\end{aligned}$$

Note that N_λ is much smaller than $W_0^{1,p}(\Omega)$ and contains the nontrivial weak solutions of (P_λ). It is possible for $\varphi_\lambda|_{N_\lambda}$ to exhibit properties which fail globally. One such property is identified in the next proposition.

Proposition 3.1: *If hypotheses H_0 hold, then $\varphi_\lambda|_{N_\lambda}$ is coercive.*

Proof: Let $u \in N_\lambda$. From the definition of the Nehari manifold we have

$$-\frac{1}{r}\int_\Omega \xi(x)|\nabla u|^p\,dx + \frac{1}{r}\int_\Omega a(x)|u|^{1-\gamma}\,dx = -\frac{\lambda}{r}\|u\|_r^r. \tag{2}$$

From (2) and hypotheses H_0 we obtain

$$\begin{aligned}
\varphi_\lambda(u) &= \left[\frac{1}{p}-\frac{1}{r}\right]\int_\Omega \xi(x)|\nabla u|^p\,dx - \left[\frac{1}{1-\gamma}-\frac{1}{r}\right]\int_\Omega a(x)|u|^{1-\gamma}\,dx\\
&\geq \left[\frac{1}{p}-\frac{1}{r}\right]\xi_0\|u\|^p - \left[\frac{1}{1-\gamma}-\frac{1}{r}\right]\int_\Omega a(x)|u|^{1-\gamma}\,dx\\
&\geq c_1\|u\|^p - c_2\|u\|^{1-\gamma}
\end{aligned} \tag{3}$$

for some $c_1, c_2 > 0$, where we have used Theorem 13.17 of Hewitt and Stromberg [12, p.196], the fact that $1-\gamma < 1 < p$ and the Sobolev embedding theorem. From (3) it is clear that $\varphi_\lambda|_{N_\lambda}$ is coercive. ■

Let $m_\lambda^+ = \inf_{N_\lambda^+} \varphi_\lambda$.

Proposition 3.2: *If hypotheses H_0 hold, then $m_\lambda^+ < 0$.*

Proof: From the definition of N_λ^+, we have, for $u \in N_\lambda^+$,

$$\lambda \|u\|_r^r < \frac{p+\gamma-1}{r+\gamma-1} \int_\Omega \xi(x)|\nabla u|^p \, \mathrm{d}x. \tag{4}$$

Moreover, since $u \in N_\lambda^+ \subseteq N_\lambda$, it holds

$$-\frac{1}{1-\gamma} \int_\Omega a(x)|u|^{1-\gamma} \, \mathrm{d}x = -\frac{1}{1-\gamma} \int_\Omega \xi(x)|\nabla u|^p \, \mathrm{d}x + \frac{\lambda}{1-\gamma} \|u\|_r^r. \tag{5}$$

Applying (4), (5), hypotheses H_0 and recalling $0 < \gamma < 1 < p < r$, we get for $u \in N_\lambda^+$

$$\begin{aligned}
\varphi_\lambda(u) &= \left[\frac{1}{p} - \frac{1}{1-\gamma}\right] \int_\Omega \xi(x)|\nabla u|^p \, \mathrm{d}x - \lambda \left[\frac{1}{r} - \frac{1}{1-\gamma}\right] \|u\|_r^r \\
&< \left[\frac{-(p+\gamma-1)}{p(1-\gamma)} + \frac{r+\gamma-1}{r(1-\gamma)} \cdot \frac{p+\gamma-1}{r+\gamma-1}\right] \int_\Omega \xi(x)|\nabla u|^p \, \mathrm{d}x \\
&= \frac{p+\gamma-1}{1-\gamma} \left[\frac{1}{r} - \frac{1}{p}\right] \int_\Omega \xi(x)|\nabla u|^p \, \mathrm{d}x \\
&< 0.
\end{aligned}$$

Therefore, $\varphi_\lambda|_{N_\lambda^+} < 0$ and so $m_\lambda^+ < 0$. ∎

Proposition 3.3: *If hypotheses H_0 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ we have $N_\lambda^0 = \emptyset$.*

Proof: We argue indirectly. So, suppose that for every $\lambda^* > 0$ there exists $\lambda \in (0, \lambda^*)$ such that $N_\lambda^0 \neq \emptyset$. Hence, given $\lambda > 0$, we can find $u \in N_\lambda$ such that

$$(p+\gamma-1) \int_\Omega \xi(x)|\nabla u|^p \, \mathrm{d}x = \lambda(r+\gamma-1)\|u\|_r^r. \tag{6}$$

Moreover, since $u \in N_\lambda$, one has

$$\begin{aligned}
&(r+\gamma-1) \int_\Omega \xi(x)|\nabla u|^p \, \mathrm{d}x - (r+\gamma-1) \int_\Omega a(x)|u|^{1-\gamma} \, \mathrm{d}x \\
&\quad = \lambda(r+\gamma-1)\|u\|_r^r.
\end{aligned} \tag{7}$$

Subtracting (6) from (7) results in

$$(r-p) \int_\Omega \xi(x)|\nabla u|^p \, \mathrm{d}x = (r+\gamma-1) \int_\Omega a(x)|u|^{1-\gamma} \, \mathrm{d}x.$$

Hence, by hypotheses H_0,

$$(r-p)\xi_0 \|u\|^p \le (r+\gamma-1)c_3 \|u\|^{1-\gamma}$$

for some $c_3 > 0$. This implies

$$\|u\|^{p+\gamma-1} \le c_4 \tag{8}$$

for some $c_4 > 0$.

On the other hand, from (6), hypotheses H_0 and the Sobolev embedding theorem, we obtain

$$\|u\|^p \leq \lambda c_5 \|u\|^r$$

for some $c_5 > 0$ and thus,

$$\left[\frac{1}{\lambda c_5}\right]^{(1/(r-p))} \leq \|u\|.$$

We let $\lambda \to 0^+$ and see that $\|u\| \to \infty$, contradicting (8). Therefore, we can find $\lambda^* > 0$ such that $N_\lambda^0 = \emptyset$ for all $\lambda \in (0, \lambda^*)$. ∎

Proposition 3.4: *If hypotheses H_0 hold, then there exists $\hat{\lambda}^* \in (0, \lambda^*]$ such that for every $\lambda \in (0, \hat{\lambda}^*)$, there exists $u^* \in N_\lambda^+$ such that*

$$\varphi_\lambda(u^*) = m_\lambda^+ = \inf_{N_\lambda^+} \varphi_\lambda$$

and $u^*(x) \geq 0$ for a. a. $x \in \Omega$.

Proof: Let $\{u_n\}_{n\geq 1} \subseteq N_\lambda^+$ be a minimizing sequence, that is,

$$\varphi_\lambda(u_n) \searrow m_\lambda^+ < 0 \quad \text{as } n \to \infty. \tag{9}$$

Since $N_\lambda^+ \subseteq N_\lambda$, from Proposition 3.1, we infer that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u^* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u^* \quad \text{in } L^r(\Omega). \tag{10}$$

From (9) and $u_n \xrightarrow{w} u^*$ in $W_0^{1,p}(\Omega)$ we have

$$\varphi_\lambda(u^*) \leq \liminf_{n\to\infty} \varphi_\lambda(u_n) < 0 = \varphi_\lambda(0).$$

Hence, $u^* \neq 0$.

We consider the fibering function $\psi_{u^*}\colon [0, \infty) \to \mathbb{R}$ defined by

$$\psi_{u^*}(t) = \varphi_\lambda(tu^*) \quad \text{for all } t \geq 0.$$

Moreover, let $\eta_{u^*}\colon (0, \infty) \to \mathbb{R}$ be the function defined by

$$\eta_{u^*}(t) = t^{p-r} \int_\Omega \xi(x)|\nabla u^*|^p \,\mathrm{d}x - t^{-\gamma-r+1} \int_\Omega a(x)|u^*|^{1-\gamma} \,\mathrm{d}x \quad \text{for all } t > 0.$$

Note that as $t \to 0^+$, then $\eta_{u^*}(t) \to -\infty$, since $r - p < r + \gamma - 1$ and $a(x) > 0$ for a. a. $x \in \Omega$, see H_0. Also, $\eta_{u^*}(t) \to 0$ as $t \to +\infty$ and $\eta_{u^*}(t) > 0$ for

$$t > \left[\frac{\int_\Omega a(x)|u^*|^{1-\gamma} \,\mathrm{d}x}{\int_\Omega \xi(x)|\nabla u^*|^p \,\mathrm{d}x}\right]^{(1/(p+\gamma-1))} = \hat{t} > 0.$$

Therefore, we can find $t_0 > \hat{t}$ such that

$$\eta_{u^*}(t_0) = \max_{t>0} \eta_{u^*}.$$

This maximizer is unique and it is given by the solution of

$$\eta'_{u^*}(t) = 0.$$

Hence,

$$t_0 = \left[\frac{(r+\gamma-1)\displaystyle\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x}{(r-p)\displaystyle\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x} \right]^{(1/(p+\gamma-1))}.$$

We see that

$$tu^* \in N_\lambda \quad \text{if and only if} \quad \eta_{u^*}(t) = \lambda\|u^*\|_r^r > 0.$$

Let $\hat{\lambda}^* \in (0, \lambda^*]$ such that

$$\eta_{u^*}(t_0) > \lambda\|u^*\|_r^r \quad \text{for all } \lambda \in (0, \hat{\lambda}^*].$$

We can find $t_1 < t_0 < t_2$ such that

$$\eta_{u^*}(t_1) = \lambda\|u^*\|_r^r = \eta_{u^*}(t_2) \quad \text{and} \quad \eta'_{u^*}(t_2) < 0 < \eta'_{u^*}(t_1). \tag{11}$$

In this proof we will only use t_1, we mention the existence of t_2 as above since it will be needed in the sequel when we will minimize over N_λ^-.

Note that $\psi_{u^*} \in C^2(0, \infty)$. Therefore,

$$\psi'_{u^*}(t_1) = t_1^{p-1}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x - t_1^{-\gamma}\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x - \lambda t_1^{r-1}\|u^*\|_r^r$$

and

$$\begin{aligned}\psi''_{u^*}(t_1) = (p-1)t_1^{p-2}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x + \gamma t_1^{-\gamma-1}\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x \\ - (r-1)\lambda t_1^{r-2}\|u^*\|_r^r.\end{aligned} \tag{12}$$

From (11) we have

$$t_1^{p-r}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x - \lambda\|u^*\|_r^r = t_1^{-\gamma-r+1}\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x,$$

which implies that

$$t_1^{p-2}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x - \lambda t_1^{r-2}\|u^*\|_r^r = t_1^{-\gamma-1}\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x. \tag{13}$$

We will now apply (13) in (12) and obtain

$$\begin{aligned}\psi''_{u^*}(t_1) &= [p+\gamma-1]t_1^{p-2}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x - (r+\gamma-1)\lambda t_1^{r-2}\|u^*\|_r^r \\ &= t_1^{-2}\left[(p+\gamma-1)t_1^p\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x - (r+\gamma-1)\lambda t_1^r\|u^*\|_r^r\right].\end{aligned} \tag{14}$$

But using (13) in (12) gives

$$
\begin{aligned}
\psi''_{u^*}(t_1) &= (p-1)t_1^{p-2}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x + \gamma t_1^{-\gamma-1}\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x\\
&\quad -(r-1)t_1^{r-2}\left[t_1^{p-r}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x - t_1^{-\gamma-r+1}\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x\right]\\
&= (p-r)t_1^{p-2}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x + (r+\gamma-1)t_1^{-\gamma-1}\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x\\
&= t_1^{r-1}\eta'_{u^*}(t_1) > 0,
\end{aligned}
\tag{15}
$$

because of (11).

From (14) and (15) it follows that

$$
(p+\gamma-1)t_1^p\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x - (r+\gamma-1)\lambda t_1^r\|u^*\|_r^r > 0,
$$

which implies

$$
t_1u^* \in N_\lambda^+, \quad \lambda \in (0,\hat{\lambda}^*]. \tag{16}
$$

Suppose that

$$
\liminf_{n\to\infty}\int_\Omega \xi(x)|\nabla u_n|^p\,\mathrm{d}x > \int_\Omega \xi(x)|\nabla u^*|\,\mathrm{d}x. \tag{17}
$$

Applying (10), (11) and (17) we get

$$
\begin{aligned}
\liminf_{n\to\infty}\psi'_{u_n}(t_1) &= \liminf_{n\to\infty}\left[t_1^{p-1}\int_\Omega \xi(x)|\nabla u_n|^p\,\mathrm{d}x - t_1^{-\gamma}\int_\Omega a(x)|u_n|^{1-\gamma}\,\mathrm{d}x - \lambda t_1^{r-1}\|u_n\|_r^r\right]\\
&> t_1^{p-1}\int_\Omega \xi(x)|\nabla u^*|^p\,\mathrm{d}x - t_1^{-\gamma}\int_\Omega a(x)|u^*|^{1-\gamma}\,\mathrm{d}x - \lambda t_1^{r-1}\|u^*\|_r^r\\
&= \psi'_{u^*}(t_1)\\
&= t_1^{r-1}\left[\eta_{u^*}(t_1) - \lambda\|u^*\|_r^r\right] = 0.
\end{aligned}
\tag{18}
$$

From (18) we see that there exists $n_0 \in \mathbb{N}$ such that

$$
\psi'_{u_n}(t_1) > 0 \quad \text{for all } n \geq n_0. \tag{19}
$$

Recall that $u_n \in N_\lambda^+ \subseteq N_\lambda$ and $\psi'_{u_n}(t) = t^r\eta_{u_n}(t)$. Hence

$$
\psi'_{u_n}(t) < 0 \quad \text{for all } t \in (0,1) \quad \text{and} \quad \psi'_{u_n}(1) = 0.
$$

Then, by (19), it follows $t_1 > 1$.

Since ψ_{u^*} is decreasing on $(0, t_1]$, we have

$$\varphi_\lambda(t_1 u^*) \leq \varphi_\lambda(u^*) < m_\lambda^+. \tag{20}$$

But recall that $t_1 u^* \in N_\lambda^+$ because of (16). So, by (20), we obtain

$$m_\lambda^+ \leq \varphi_\lambda(t_1 u^*) < m_\lambda^+,$$

a contradiction. This proves that $u_n \to u^*$ in $W_0^{1,p}(\Omega)$, see Papageorgiou and Winkert [13, p.225], and so, with regards to (9),

$$\varphi_\lambda(u_n) \to \varphi_\lambda(u^*) = m_\lambda^+ < 0.$$

We know that $u_n \in N_\lambda^+$ for all $n \in \mathbb{N}$. This implies

$$(p + \gamma - 1) \int_\Omega \xi(x)|\nabla u_n|^p \,\mathrm{d}x > \lambda(r + \gamma - 1)\|u_n\|_r^r \quad \text{for all } n \in \mathbb{N}.$$

Therefore

$$(p + \gamma - 1) \int_\Omega \xi(x)|\nabla u^*|^p \,\mathrm{d}x \geq \lambda(r + \gamma - 1)\|u^*\|_r^r. \tag{21}$$

On account of Proposition 3.3, since $\lambda \in (0, \hat{\lambda}^*]$, we cannot have equality in (21). Therefore $u^* \in N_\lambda^+$ and finally we have

$$m_\lambda^+ = \varphi_\lambda(u^*) \quad \text{and} \quad u^* \in N_\lambda^+.$$

Since we can always replace u^* by $|u^*|$, we may assume that $u^* \geq 0$ with $u^* \neq 0$. ■

The next lemma is inspired by Lemma 3 of Sun et al. [3]. In what follows we denote by $B_\varepsilon(0)$ the open ε-ball in $W_0^{1,p}(\Omega)$ centered at the origin, that is,

$$B_\varepsilon(0) = \left\{ u \in W_0^{1,p}(\Omega) \colon \|u\| < \varepsilon \right\}.$$

Lemma 3.5: *If hypotheses H_0 hold and $u \in N_\lambda^+$, then there exist $\varepsilon > 0$ and a continuous function $\vartheta : B_\varepsilon(0) \to \mathbb{R}_+$ such that*

$$\vartheta(0) = 1 \quad \text{and} \quad \vartheta(y)(u + y) \in N_\lambda^\pm \quad \textit{for all } y \in B_\varepsilon(0).$$

Proof: We do the proof only for N_λ^+, the proof for N_λ^- works in the same way. So, let $L\colon W_0^{1,p}(\Omega) \times (0, \infty) \to \mathbb{R}$ be defined by

$$L(y, t) = t^{p+\gamma-1} \int_\Omega \xi(x)|\nabla(u + y)|^p \,\mathrm{d}x - \int_\Omega a(x)|u + y|^{1-\gamma} \,\mathrm{d}x - \lambda t^{r+\gamma-1}\|u + y\|_r^r.$$

Since $u \in N_\lambda^+ \subseteq N_\lambda$, one has $L(0, 1) = 0$. Moreover, because $u \in N_\lambda^+$, it holds

$$L_t'(0, 1) = (p + \gamma - 1) \int_\Omega \xi(x)|\nabla u|^p \,\mathrm{d}x - \lambda(r + \gamma - 1)\|u\|_r^r > 0.$$

Then, by the implicit function theorem, see Gasiński and Papageorgiou [14, p.481], we can find $\varepsilon > 0$ and a continuous map $\vartheta : B_\varepsilon(0) \to \mathbb{R}_+$ such that

$$\vartheta(0) = 1 \quad \text{and} \quad \vartheta(y)(u + y) \in N_\lambda \quad \text{for all } y \in B_\varepsilon(0).$$

Choosing $\varepsilon > 0$ even smaller if necessary, we can have

$$\vartheta(0) = 1 \quad \text{and} \quad \vartheta(y)(u + y) \in N_\lambda^+ \quad \text{for all } y \in B_\varepsilon(0).$$ ■

Proposition 3.6: *If hypotheses H_0 hold, $\lambda \in (0, \hat{\lambda}^*]$ and $h \in W_0^{1,p}(\Omega)$, then we can find $b > 0$ such that*

$$\varphi_\lambda(u^*) \leq \varphi(u^* + th) \quad \textit{for all } t \in [0, b].$$

Proof: We consider the function $\mu_h \colon [0, \infty) \to \mathbb{R}$ defined by

$$\begin{aligned} \mu_h(t) = (p-1) \int_\Omega \xi(x) |\nabla u^* + t\nabla h|^p \, \mathrm{d}x \\ + \gamma \int_\Omega a(x) |u^* + th|^{1-\gamma} \, \mathrm{d}x - \lambda(r-1)\|u^*\|_r^r. \end{aligned} \tag{22}$$

Recall that $u^* \in N_\lambda^+ \subseteq N_\lambda$, see Proposition 3.4. Thus, we have

$$\gamma \int_\Omega \xi(x)|u^*|^{1-\gamma} \, \mathrm{d}x = \gamma \int_\Omega \xi(x)|\nabla u^*|^p \, \mathrm{d}x - \lambda\gamma \|u^*\|_r^r \tag{23}$$

and

$$(p + \gamma - 1) \int_\Omega \xi(x)|\nabla u^*|^p \, \mathrm{d}x - \lambda(r + \gamma - 1)\|u\|_r^r > 0. \tag{24}$$

Combining (22), (23) and (24) we obtain that

$$\mu_h(0) > 0. \tag{25}$$

The function μ_h is continuous. So, we can find $b_0 > 0$ such that

$$\mu_h(t) > 0 \quad \text{for all } t \in (0, b_0),$$

see (25). Lemma 3.5 implies that for every $t \in [0, b_0)$, we can find $\hat{\vartheta}(t) > 0$ such that

$$\hat{\vartheta}(t)(u^* + th) \in N_\lambda^+ \quad \text{and} \quad \hat{\vartheta}(t) \to 1 \text{ as } t \to 0^+. \tag{26}$$

Taking (26) into account we finally reach that

$$\begin{aligned} m_\lambda^+ = \varphi_\lambda(u^*) &\leq \varphi_\lambda(\hat{\vartheta}(t)(u^* + th)) \quad \text{for all } t \in [0, b_0) \\ &\leq \varphi_\lambda(u^* + th) \quad \text{for all } t \in [0, b) \text{ with } b \leq b_0. \end{aligned}$$

■

The next proposition shows that N_λ^+ is a natural constraint for the functional φ_λ, see Papageorgiou et al. [15, p.425].

Proposition 3.7: *If hypotheses H_0 hold and $\lambda \in (0, \hat{\lambda}^*)$, then u^* is a weak solution of problem (P_λ).*

Proof: Let $h \in W_0^{1,p}(\Omega)$. From Proposition 3.6 we know that

$$0 \leq \varphi_\lambda(u^* + th) - \varphi_\lambda(u^*) \quad \text{for all } 0 < t < b.$$

This means

$$\begin{aligned} &\frac{1}{1-\gamma} \int_\Omega a(x) \left[|u^* + th|^{1-\gamma} - |u^*|^{1-\gamma} \right] \mathrm{d}x \\ &\leq \frac{1}{p} \int_\Omega \xi(x) \left(|\nabla(u^* + th)|^p - |\nabla u^*|^p \right) \mathrm{d}x - \frac{\lambda}{r} \left[\|u^* + th\|_r^r - \|u^*\|_r^r \right]. \end{aligned}$$

Multiplying by $(1/t)$ and letting $t \to 0^+$ gives

$$\int_\Omega a(x)(u^*)^{-\gamma} h \,\mathrm{d}x \le \int_\Omega \xi(x)|\nabla u^*|^{p-2}\nabla u^* \cdot \nabla h \,\mathrm{d}x - \lambda \int_\Omega (u^*)^{r-1} h \,\mathrm{d}x$$

for all $h \in W_0^{1,p}(\Omega)$. Hence,

$$\int_\Omega \xi(x)|\nabla u^*|^{p-2}\nabla u^* \cdot \nabla h \,\mathrm{d}x = \int_\Omega a(x)(u^*)^{-\gamma} h \,\mathrm{d}x + \lambda \int_\Omega (u^*)^{r-1} h \,\mathrm{d}x$$

for all $h \in W_0^{1,p}(\Omega)$. Thus, u^* is a weak solution of (P_λ). ■

Now we are ready to generate the first positive solution of problem (P_λ).

Proposition 3.8: *If hypotheses H_0 hold and $\lambda \in (0, \hat{\lambda}^*)$, then problem (P_λ) admits a positive solution $u^* \in W_0^{1,p}(\Omega)$ such that $u \in L^\infty(\Omega)$, $u^*(x) > 0$ for a. a. $x \in \Omega$ and $\varphi_\lambda(u^*) < 0$.*

Proof: According to Proposition 3.4 there exists $u^* \in W_0^{1,p}(\Omega)$ such that

$$u^* \in N_\lambda^+ \quad \text{and} \quad m_\lambda^+ = \varphi_\lambda(u^*) < 0, \quad u^* \ge 0.$$

From Proposition 3.7 we know that u^* is a weak solution of problem (P_λ).

From Giacomoni et al. [5, Lemma A.6, p.142] we have that $u^* \in L^\infty(\Omega)$. Furthermore, the Harnack inequality, see Pucci and Serrin [2, p.163] implies that

$$u^*(x) > 0 \quad \text{for a. a. } x \in \Omega.$$ ■

Now we start looking for a second positive solution. To this end, we will use the manifold N_λ^-.

Proposition 3.9: *If hypotheses H_0 hold, then there exists $\hat{\lambda}_0^* \in (0, \hat{\lambda}^*]$ such that $\varphi_\lambda|_{N_\lambda^-} \ge 0$ for all $0 < \lambda \le \hat{\lambda}_0^*$.*

Proof: Let $u \in N_\lambda$. From the definition of N_λ^- we have

$$(p+\gamma-1)\int_\Omega \xi(x)|\nabla u|^p \,\mathrm{d}x < \lambda(r+\gamma-1)\|u\|_r^r,$$

which implies

$$(p+\gamma-1)\xi_0\|\nabla u\|_p^p < \lambda(r+\gamma-1)\|u\|_r^r.$$

Then, by the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, it follows

$$(p+\gamma-1)\xi_0 c_5\|u\|_r^p < \lambda(r+\gamma-1)\|u\|_r^r$$

for some $c_5 > 0$. Therefore

$$\left[\frac{(p+\gamma-1)\xi_0 c_5}{\lambda(r+\gamma-1)}\right]^{(1/((r-p))} \le \|u\|_r. \tag{27}$$

Suppose that the result of the proposition is not true. This means that for every $\lambda > 0$ there exists $u \in N_\lambda^-$ such that $\varphi_\lambda(u) < 0$, that is,

$$\frac{1}{p}\int_\Omega \xi(x)|\nabla u|^p \,\mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma} \,\mathrm{d}x - \frac{\lambda}{r}\|u\|_r^r < 0. \tag{28}$$

On the other hand, since $u \in N_\lambda^- \subseteq N_\lambda$, we have

$$\int_\Omega \xi(x)|\nabla u|^p \,\mathrm{d}x = \int_\Omega a(x)|u|^{1-\gamma} \,\mathrm{d}x + \lambda \|u\|_r^r. \tag{29}$$

Using (29) in (28) yields

$$\left[\frac{1}{p} - \frac{1}{1-\gamma}\right] \int_\Omega a(x)|u|^{1-\gamma} \,\mathrm{d}x + \lambda \left[\frac{1}{p} - \frac{1}{r}\right] \|u\|_r^r < 0,$$

which implies

$$\lambda \frac{r-p}{pr} \|u\|_r^r \leq \frac{p+\gamma-1}{p(1-\gamma)} \int_\Omega a(x)|u|^{1-\gamma} \,\mathrm{d}x \leq \frac{p+\gamma-1}{p(1-\gamma)} c_6 \|u\|_r^{1-\gamma}$$

for some $c_6 > 0$. Hence

$$\|u\|_r \leq \left[\frac{(p+\gamma-1)rc_6}{\lambda(1-\gamma)(r-p)}\right]^{(1/((r+\gamma-1))}$$

and so

$$\|u\|_r \leq c_7 \left(\frac{1}{\lambda}\right)^{(1/(r+\gamma-1))} \tag{30}$$

for some $c_7 > 0$.

Now we use (30) in (27) and obtain

$$c_8 \left(\frac{1}{\lambda}\right)^{(1/(r-p))} \leq c_7 \left(\frac{1}{\lambda}\right)^{(1/(r+\gamma-1))} \quad \text{with } c_8 = \left[\frac{(p+\gamma-1)\xi_0}{r+\gamma-1}\right]^{(1/((r-p))} > 0.$$

This implies

$$c_9 \leq \lambda^{\frac{p+\gamma-1}{(r+\gamma-1)(r-p)}} \quad \text{with } c_9 = \frac{c_8}{c_7} > 0.$$

Letting $\lambda \to 0^+$ leads to a contradiction. So, we can find $0 < \hat{\lambda}_0^* \leq \hat{\lambda}^*$ such that $\varphi_\lambda|_{N_\lambda^-} \geq 0$ for all $\lambda \in (0, \hat{\lambda}_0^*]$. ■

Now we minimize φ_λ on the manifold N_λ^-.

Proposition 3.10: *If hypotheses H_0 hold and $\lambda \in (0, \lambda_0^*)$, then we can find $v^* \in N_\lambda^-$ with $v^* \geq 0$ such that*

$$m_\lambda^- = \inf_{N_\lambda^-} \varphi_\lambda = \varphi_\lambda(v^*).$$

Proof: The proof of the proposition is the same as that of Proposition 3.4. Only now as we already hinted in that proof, we use the point $t_2 > t_0$ for which we have

$$\eta_{v^*}(t_2) = \lambda \|v^*\|_r^r \quad \text{and} \quad \eta'_{v^*}(t_2) < 0,$$

see (11). Then we conclude that

$$v^* \in N_\lambda^-, \quad v^* \geq 0, \quad m_\lambda^- = \varphi_\lambda(v^*). \qquad ■$$

Applying Lemma 3.5 and reasoning as in the proofs of Propositions 3.6 and 3.7 we show that N_λ^- is a natural constraint for the energy functional φ_λ as well.

Proposition 3.11: *If hypotheses H_0 hold and $\lambda \in (0, \hat{\lambda}_0^*)$, then v^* is a weak solution of problem (P_λ).*

Therefore, we have a second positive solution $v^* \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and by Harnack's inequality we have $v^*(x) > 0$ for a. a. $x \in \Omega$.

Finally, we can state the following multiplicity theorem for problem (P_λ).

Theorem 3.12: *If hypotheses H_0 hold, then there exists $\hat{\lambda}_0^* > 0$ such that for all $\lambda \in (0, \hat{\lambda}_0^*)$, problem (P_λ) has at least two positive solutions*

$$u^*, v^* \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad u^*(x) > 0,\ v^*(x) > 0 \quad \text{for a. a. } x \in \Omega$$

and

$$\varphi_\lambda(u^*) < 0 < \varphi_\lambda(v^*).$$

Remark 3.13: It is an interesting open problem whether the multiplicity theorem above holds if we assume that

$$\xi \in L^\infty(\Omega) \quad \text{and} \quad \xi(x) > 0 \quad \text{for a. a. } x \in \Omega,$$

but not necessarily bounded away from zero.

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References

[1] Lieberman GM. The natural generalization of the natural conditions of Ladyzhenskaya and Ural′ tseva for elliptic equations. Commun Partial Differ Equ. 1991;16(2–3):311–361.
[2] Pucci P, Serrin J. The maximum principle. Basel: Birkhäuser Verlag; 2007.
[3] Sun Y, Wu S, Long Y. Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. J Differ Equ. 2001;176(2):511–531.
[4] Haitao Y. Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem. J Differ Equ. 2003;189(2):487–512.
[5] Giacomoni J, Schindler I, Takáč P. Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation. Ann Sc Norm Super Pisa Cl Sci (5). 2007;6(1):117–158.
[6] Papageorgiou NS, Smyrlis G. A bifurcation-type theorem for singular nonlinear elliptic equations. Methods Appl Anal. 2015;22(2):147–170.
[7] Papageorgiou NS, Winkert P. Singular p-Laplacian equations with superlinear perturbation. J Differ Equ. 2019;266(23):1462–1487.
[8] Perera K, Zhang Z. Multiple positive solutions of singular p-Laplacian problems by variational methods. Bound Value Probl. 2006;2005(3):377–382.
[9] Leonardi S, Papageorgiou NS. Positive solutions for nonlinear Robin problems with indefinite potential and competing nonlinearities. Positivity. 2019. doi:10.1007/s11117-019-00681-5

[10] Leonardi S, Papageorgiou NS. On a class of critical Robin problems. Forum Math. 2019. doi:10.1515/forum-2019-0160
[11] Gasiński L, Papageorgiou NS. Exercises in analysis. Part 2: nonlinear analysis. Heidelberg: Springer; 2016.
[12] Hewitt E, Stromberg K. Real and abstract analysis. New York (NY): Springer-Verlag; 1965.
[13] Papageorgiou NS, Winkert P. Applied nonlinear functional analysis. An introduction. Berlin: De Gruyter; 2018.
[14] Gasiński L, Papageorgiou NS. Nonlinear analysis. Boca Raton (FL): Chapman & Hall/CRC; 2006.
[15] Papageorgiou NS, Rădulescu VD, Repovš DD. Nonlinear analysis – theory and methods. Cham: Springer; 2019.